A panorama of algebraic structures

Abdenacer MAKHLOUF
University of Haute Alsace [1]

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Abstract

The main purpose of this course is to provide an introduction to non-associative algebras and their dualization, together with their applications in various domains of Mathematics and Physics. Moreover we provide some related key structures, working as invariants and playing an important role like representations, cohomology and deformations.

The course gives detailed expositions of fundamental concepts along with an introduction to recent developments in the topics using these fundamental tools.

Contents

[1] Abdenacer.Makhlouf@uha.fr

1 Associative algebras

Definition 1.1 *Let $\mathbb{K}$ be a field of characteristic 0 and A be a vector space over $\mathbb{K}$. An associative $\mathbb{K}$-algebra structure on A (or simply an algebra) is given by a* *bilinear map $A \times A \to A$ satisfying the following conditions*

$$(A1) \qquad \forall x, y, z \in A \quad \mu(\mu(x, y), z) - \mu(x, \mu(y, z)) = 0.$$

The multiplication μ has a unit. That is, there exists $e \in A$ such that

$$(A2) \qquad \forall x \in A \quad \mu(x, e) = \mu(e, x) = x$$

When there is no ambiguity one uses 1 or $1_{\mathcal{A}}$ instead of e.

An algebra is said to be finite dimensional or infinite dimensional according to whether the space A is finite dimensional or infinite dimensional. The dimension of the vector space is called the dimension of the algebra.

Remark 1.1 1. *In general, let R be a commutative ring. A R-algebra is a unitary R-module $\mathcal{A}$ on which we define a bilinear map*

$$\begin{aligned} \mathcal{A} \times \mathcal{A} &\to \mathcal{A} \\ (x, y) &\to x \cdot y \end{aligned}$$

satisfying the associativity and the existence of the unit 1_A.

2. *Every R-algebra is a unitary ring, the bilinearity of the multiplication is equivalent to the right and left distributivity*

$$(xy)a = x(ya) = (xa)y \quad \forall x, y \in \mathcal{A}, a \in R$$

Conversely, if $\mathcal{A}$ is a unitary ring and a right R-module satisfying the previous condition then $\mathcal{A}$ is an R-algebra.

3. *Let $\mathcal{A}$ be an R-algebra. The following map*

$$\begin{aligned} R &\to \mathbf{Z}(\mathcal{A}) = \{x \in \mathcal{A} : x \cdot y = y \cdot x \; \forall y \in A\} \\ a &\to 1_A a \end{aligned}$$

is a ring homomorphism.

Conversely, if $\mathcal{A}$ is a ring, every homomorphism from R to $\mathbf{Z}(\mathcal{A})$ induces on $\mathcal{A}$ an R-module structure which transforms $\mathcal{A}$ to an R-algebra.

The previous remark allows to give an alternative definition : An algebra $\mathcal{A}$ is a ring $\mathcal{A}$ and a ring homomorphism

$$\begin{aligned} \varphi : \quad R &\to \mathbf{Z}(\mathcal{A}) \\ a &\to 1_{\mathcal{A}} a. \end{aligned}$$

4. *If the map $a \to 1_{\mathcal{A}} a$ is injective (the R-module is said faithful) then R is identified to a subring of* $\mathbf{Z}(\mathcal{A})$ $(Ker\varphi)$. *Using this identification* $(xa = ax)$, *$\mathcal{A}$ becomes a left R-module.*

5. *Using Hopf algebras language and a same notation for the multiplication, an associative algebra A is a triple (A, μ, η) where μ and η are linear maps:*

$$\mu : \begin{array}{c} A \otimes A \to A \\ x \otimes y \to \mu(x \otimes y) \end{array} \qquad and \qquad \eta : \begin{array}{c} \mathbb{K} \to A \\ a \to \eta(a) = a \cdot \eta(1) \end{array}$$

satisfying the associativity condition and the existence of unit. These conditions are expressed by the following commuting diagrams

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes id} & A \otimes A \\ \downarrow^{id \otimes \mu} & & \downarrow^{\mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array} \qquad \begin{array}{ccccc} \mathbb{K} \otimes A & \xrightarrow{\eta \otimes id} & A \otimes A & \xleftarrow{id \otimes \eta} & A \otimes \mathbb{K} \\ & \searrow^{\cong} & \downarrow^{\mu} & \swarrow^{\cong} & \\ & & A & & \end{array}$$

When there is no ambiguity the multiplication is denoted by a dot "$\cdot$" or just by concatenation of the elements.

1.1 Examples

1. $\mathbb{C}^n$ and $\mathbb{R}^n$
2. The polynomial rings $\mathbb{K}[x]$ and $\mathbb{K}[x_1, ..., x_n]$.
3. The matrix algebra $M_n(\mathbb{K})$. The multiplication is a matrix multiplication. It gives an example of a noncommutative algebra.
4. The group algebra. Let G be a group with the elements $\{\alpha_i\}_i$. The vector space $\mathbb{K}G$ is generated by the basis $\{e_{\alpha_i}\}_i$. The multiplication is defined by $e_{\alpha_i} \dot{e}_{\alpha_j} = e_{\alpha_i \alpha_j}$
5. Tensor product of algebras. Let A and B be two algebras over $\mathbb{K}$ We define the tensor product $A \otimes B$ as follows :

   the bilinearity implies that for $u_1, u_2 \in A$ $v_1, v_2 \in B$, $\alpha, \beta \in \mathbb{K}$

   $(\alpha u_1 + \beta u_2) \otimes v_1 = \alpha u_1 \otimes v_1 + \beta u_2 \otimes v_1$

   $u_1 \otimes (\alpha v_1 + \beta v_2) = \alpha u_1 \otimes v_1 + \beta u_1 \otimes v_2$

   The multiplication on the tensor product A and B is

   $(u_1 \otimes v_1) \cdot_{A \otimes B} (u_2 \otimes v_2) = u_1 \cdot_A u_2 \otimes v_1 \cdot_B v_2$

1.2 Morphism of algebras

If R is a ring, the class of R-algebra forms a category where the morphisms are simultaneously the ring and module homomorphisms preserving the unit. They are called morphisms of algebras.

Let (V, μ, η) and (V', μ', η') be two algebras. A linear map $f : V \to V'$ is a morphism of algebras if

$$\mu' \circ (f \otimes f) = f \circ \mu \qquad \text{and} \qquad f \circ \eta = \eta'$$

In particular, (V, μ, η) and (V, μ', η') are isomorphic if there exists a bijective linear map f such that

$$\mu = f^{-1} \circ \mu' \circ (f \otimes f) \qquad \text{and} \qquad \eta = f^{-1} \circ \eta'$$

2 Representations and Modules

Definition 2.1 *A representation of a $\mathbb{K}$-algebra $\mathcal{A}$ is a homomorphism T of $\mathcal{A}$ into the algebra $End(W)$ of the linear operators on a $\mathbb{K}$-vector space W.*

In other words, define a representation T is assign to every element $a \in \mathcal{A}$ a linear operator $T(a)$ in such a way that

T: A → End(W)

$$\begin{aligned} T(a+b) &= T(a) + T(b), \quad & T(\alpha a) &= \alpha T(a), \\ T(a\, b) &= T(a)T(b), \quad & T(1_{\mathcal{A}}) &= Id. \end{aligned}$$

For arbitrary $a, b \in \mathcal{A}$, $\alpha \in \mathbb{K}$ and where Id denotes The identity operator.

If the vector space W is finite dimensional then its dimension is called the dimension (or degree) of the representation. The image of the representation T forms a subalgebra of $End(W)$. If T is injective, then this subalgebra is isomorphic to the algebra $\mathcal{A}$. In this case, the representation is said to be faithful.

Theorem 2.1 (Cayley) *Every algebra admits a faithful representation. That is every algebra is isomorphic to a subalgebra of the algebra of linear operators.*

Proof 1 *It follows from the axioms of an algebra that, for any $a \in \mathcal{A}$, the map $T(a) : x \to xa$, $x \in A$, is a representation of $\mathcal{A}$. If $a \neq b$, it follows that the operators $T(a)$ and $T(b)$ are distinct and that T is faithful representation.*

The representation defined in the proof is called regular representation.

In particular, every finite dimensional algebra is isomorphic to a subalgebra of a matrix algebra. In order to view the representation just as operators in W, one needs the concept of module.

Definition 2.2 *A right module over a* $\mathbb{K}$*-algebra* $\mathcal{A}$*, or right* $\mathcal{A}$*-module, is a vector space* M *over the field* $\mathbb{K}$ *such that to every pair* $(m, a),\ m \in M,\ a \in \mathcal{A}$*, there corresponds a uniquely determined element* $ma \in M$ *satisfying the following conditions*

$$\begin{aligned} (m_1 + m_2)a &= m_1 a + m_2 a, & m(a_1 + a_2) &= ma_1 + ma_2, \\ (\alpha m)a &= m(\alpha a), & (\alpha m)a &= \alpha(ma), \\ m(ab) &= (ma)b, & m1_{\mathcal{A}} &= m. \end{aligned}$$

For arbitrary $m_1, m_2, m \in M$*,* $a_1, a_2, a \in \mathcal{A}$ *and* $\alpha \in \mathbb{K}$

Remark 2.1 *There is a correspondence between representations of an algebra* $\mathcal{A}$ *and left* A*-modules.*

In fact, let $T(a) : x \to xa,\ x \in A$ *be a representation of the algebra* $\mathcal{A}$*. define the product of the element of* W *by the elements of the algebra by putting* $wa = wT(a)$ *for any* $w \in W$ *and* $a \in \mathcal{A}$*. On the other hand, if* M *is right* $\mathcal{A}$*-module, then it follows that for a fixed* $a \in \mathcal{A}$*, the map* $T(a) : m \to ma,\ x \in A$ *is are representation of* $\mathcal{A}$*.*

A homomorphism of a right $\mathcal{A}$-module M into a right $\mathcal{A}$-module N is a linear map $f : M \to N$ for which $(ma)f = (mf)a$ for arbitrary elements $m \in M$ and $a \in \mathcal{A}$.

The set of all homomorphisms of a right $\mathcal{A}$-module M into a right $\mathcal{A}$-module N, denoted by $Hom(M, N)$ is a vector space over $\mathbb{K}$ with the natural operations.

Analogously to the concept of a right module, one can define a *left module* over the algebra $\mathcal{A}$. To left modules, there corresponds the *antirepresentations* of the algebra $\mathcal{A}$, that is the linear transformations $T : \mathcal{A} \to End(W)$ such that $T(a + b) = T(b) + T(a)$ and $T(1_{\mathcal{A}}) = Id$.

Definition 2.3 *Let* A *and* B *two algebras. A vector space* M *is an* $A - B$*-bimodule if it is a left* A*-module and right* B*-module*

$$\forall a \in A \quad \forall x \in M \quad \forall b \in B \quad a(xb) = (ax)b.$$

We write A*-bimodule when* $A = B$*.*

For example any algebra A is an A-bimodule over itself where the actions are given by the multiplication.

3 Hopf algebras

3.1 Coalgebra

A coalgebra is a triple (V, Δ, ε) where V is a $\mathbb{K}$-vector space and
$\Delta : V \to V \otimes V$ and $\varepsilon : V \to \mathbb{K}$ are linear maps satisfying the following conditions

(C1) $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$

(C2) $(id \otimes \varepsilon) \circ \Delta = id$ et $(\varepsilon \otimes id) \circ \Delta = id$

Remark 3.1 *1. (C1) expressed that Δ, the comultiplication, is coassociative and (C2) that ε is the counit.*

2. *Let (V, Δ, ε) and $(V', \Delta', \varepsilon')$ be two coalgebras. A linear map $f : V \to V'$ is a morphism of coalgebras if*

   $(f \otimes f) \circ \Delta = \Delta' \circ f$ *and* $\varepsilon = \varepsilon' \circ f$.

3. *If $V = V'$ the previous coalgebras are isomorphic if there exists a bijective linear map*
   $f : V \to V$ such that

$$\Delta' = (f \otimes f) \circ \Delta \circ f^{-1} \qquad \text{and} \qquad \varepsilon' = \varepsilon \circ f^{-1}$$

4. *The dual vector space of a coalgebra is an algebra.*

5. *The dual vector space of a finite-dimensional algebra has a coalgebra structure.*

6. *A coalgebra is said cocommutative if $\tau \circ \Delta = \Delta$ where $\tau(x \otimes y) = y \otimes x$.*

7. *The subset $I \in V$ is a coideal if $\Delta I \subseteq I \otimes V + V \otimes I$ and $\varepsilon(I) = 0$. If I is a coideal of a coalgebra $C = (V, \Delta, \varepsilon)$ then C/I is a coalgebra.*

3.2 Bialgebra

A bialgebra is a quintuple $(V, \mu, \eta, \Delta, \varepsilon)$ where

(B1) (V, μ, η) is an algebra

(B2) (V, Δ, ε) is a coalgebra

(B3) The linear maps Δ and ε are morphisms of algebras.

Remark 3.2 *The condition (B3) could be expressed by the following system :*

$$\begin{cases} \Delta(e_1) = e_1 \otimes e_1 & \textit{where } e_1 = \eta(1) \\ \Delta(x \cdot y) = \sum_{(x)(y)} x^{(1)} \cdot y^{(1)} \otimes x^{(2)} \cdot y^{(2)} \\ \varepsilon(e_1) = 1 \\ \varepsilon(x \cdot y) = \varepsilon(x)\,\varepsilon(y) \end{cases}$$

where using the Sweedler's notation $\Delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)}$*. If there is no ambiguity we denote the multiplication by a point.*

3.3 Hopf algebra

Given a bialgebra $(V, \mu, \eta, \Delta, \varepsilon)$. An endomorphism S is an antipode if it the inverse of the identity over V for the algebra $Hom(V, V)$ with the multiplication given by the convolution product defined by

$$f \star g = \mu \circ (f \otimes g)\,\Delta$$

the unit being $\eta \circ \epsilon$.

The condition may be expressed by the relation :

$$\mu \circ S \otimes Id \circ \Delta = \mu \circ Id \otimes S \circ \Delta = \eta \circ \varepsilon$$

A *Hopf algebra* is a bialgebra with an antipode. It is denoted by the sextuple $(V, \mu, \eta, \Delta, \varepsilon, S)$.

Remark 3.3

1. $S(x \cdot y) = S(y) \cdot S(x)$ *for* $x, y \in V$.
2. *S*$(\eta(1)) = \eta(1)$.
3. $(S \otimes S)\,\Delta = \Delta^{op} S$ *where* Δ^{op} *is the comultiplication for the opposite coalgebra.*
4. $\varepsilon \circ S = \varepsilon$.
5. *If the Hopf algebra is commutative or cocommutative then* $S^2 = Id$.
6. *The antipode when it exists is unique.*

4 Leibniz algebras

Definition 4.1 *A Leibniz algebra is a bilinear map on* V *satisfying*

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] \tag{4.0.1}$$

We call the previous condition Jacobi condition.

we call a homomorphism $f : V \to V$ a derivation of a Leibniz algebra $(V, [\ ,\ ])$ if it satisfies

$$f([x, y]) = [f(x), y] + [x, f(y)]$$

The condition of Leibniz algebras maybe rephrased as adjoint homomorphisms Ad_X defined by $Ad_X(Y) = [X, Y]$ are derivations.

5 Lie algebras

Definition 5.1 *A vector space $\mathfrak{g}$ over $\mathbb{K}$, with a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denoted by $(x, y) \to [x, y]$ and called bracket or commutator of x and y, is called Lie algebra if $\forall x, y, z \in \mathfrak{g}$*

(1) $[x, y] = -[y, x]$ *(skew-symmetry),*

(2) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ *(Jacobi identity).*

Remark 5.1 *Every associative algebra induces a Lie algebra with the following bracket : $[x, y] = x \cdot y - y \cdot x$.*

As example we may consider matrix algebra and its subalgebras.

6 Lie admissible algebras

The Lie-admissible algebras was introduced by A. A. Albert in 1948. Physicists attempted to introduce this structure instead of Lie algebras. For instance, the validity of Lie-Admissible algebras for free particles is well known. These algebras arise also in classical quantum mechanics as a generalization of conventional mechanics (see [?]). A nonassociative algebra $\mathcal{A}$ is said to be Lie-Admissible algebra if the bracket defined for all $x, y \in \mathcal{A}$ by

$$[x, y] = \mu(x, y) - \mu(y, x)$$

satisfies the Jacobi identity, that is

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

Since the bracket is also skewsymmetric then it defines a Lie algebra. associative and Lie algebras are Lie-Admissible.

In the following, we explore some other Lie-Admissible algebras. We introduce the following notation. Let a_μ be a trilinear map over V associated to a product μ defined by

$$a_\mu(x_1, x_2, x_3) = \mu(x_1, \mu(x_2, x_3)) - \mu(\mu(x_1, x_2), x_3)$$

We call a_μ the associator of μ.

Set

$$S(x,y,z) := a_\mu(x,y,z) + a_\mu(y,z,x) + a_\mu(z,x,y)$$

We have

$$S(x,y,z) = [\mu(x,y),z] + [\mu(y,z),x] + [\mu(z,x),y]$$

A nonassociative algebra $(A,\cdot))$ is Lie admissible if and only if

$$S(x,y,z) = S(x,z,y).$$

Let G be a subgroup of the permutations group $\mathcal{S}_3$, a binary algebra on A defined by the multiplication μ is said G-associative if

$$\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}\mu(x_{\sigma(1)},\mu(x_{\sigma(2)},x_{\sigma(3)})) - \mu(\mu(x_{\sigma(1)},x_{\sigma(2)}),x_{\sigma(3)}) = 0 \qquad (6.0.1)$$

where x_i are in A and $(-1)^{\varepsilon(\sigma)}$ is the signature of the permutation σ. The condition 6.0.1 may be written

$$\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}a_{\alpha,\mu}\circ\sigma = 0 \qquad (6.0.2)$$

where $\sigma(x_1,x_2,x_3) = (x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})$.

If μ is the multiplication of a Lie-Admissible Lie algebra then The condition 6.0.1 is equivalent to the bracket defined by

$$[x,y] = \mu(x,y) - \mu(y,x)$$

satisfies the Hom-Jacobi condition is equivalent to

$$\sum_{\sigma\in\mathcal{S}_3}(-1)^{\varepsilon(\sigma)}\mu(\alpha(x_{\sigma(1)}),\mu(x_{\sigma(2)},x_{\sigma(3)})) - \mu(\mu(x_{\sigma(1)},x_{\sigma(2)}),\alpha(x_{\sigma(3)})) = 0 \qquad (6.0.3)$$

which may be written

$$\sum_{\sigma\in\mathcal{S}_3}(-1)^{\varepsilon(\sigma)}a_{\alpha,\mu}\circ\sigma = 0 \qquad (6.0.4)$$

Let G be a subgroup of the permutations group $\mathcal{S}_3$. Then any G-associative algebra is a Lie-Admissible algebra.

Proof 2 *The skewsymmetry follows straightaway from the definition.*

We have a subgroup G in $\mathcal{S}_3$. Take the set of conjugacy class $\{gG\}_{g\in I}$ where $I\subseteq G$, and for any $\sigma_1,\sigma_2\in I, \sigma_1\neq\sigma_2 \Rightarrow \sigma_1 G\bigcap\sigma_1 G = \emptyset$. Then

$$\sum_{\sigma\in\mathcal{S}_3}(-1)^{\varepsilon(\sigma)}a_{\alpha,\mu}\circ\sigma = \sum_{\sigma_1\in I}\sum_{\sigma_2\in\sigma_1 G}(-1)^{\varepsilon(\sigma_2)}a_{\alpha,\mu}\circ\sigma_2 = 0$$

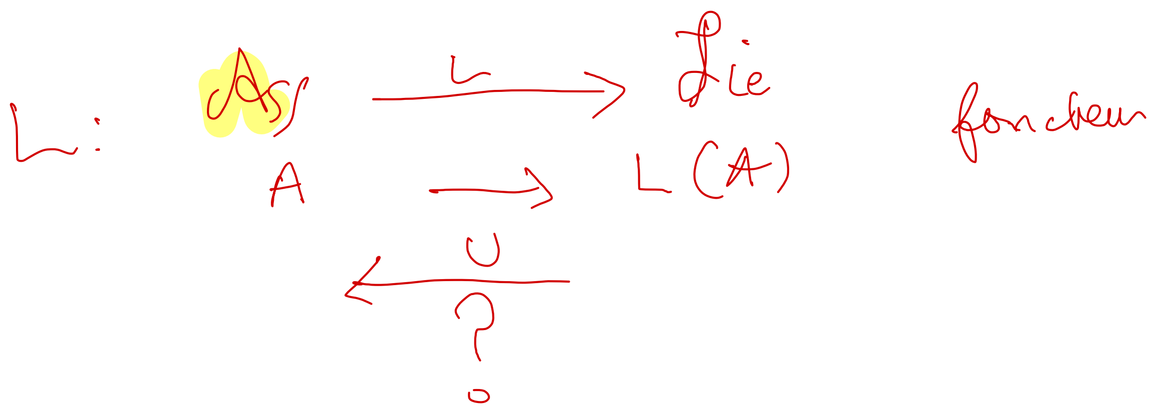
The subgroups of $\mathcal{S}_3$ are $G_1 = \{Id\},\ G_2 = \{Id, \tau_{12}\},\ G_3 = \{Id, \tau_{23}\},\ G_4 = \{Id, \tau_{13}\},\ G_5 = A_3,\ G_6 = \mathcal{S}_3$, where A_3 is the alternating group and τ_{ij} is the transposition between i and j.

We obtain the following type of Lie-admissible algebras.

1. The G_1-associative algebras are the associative algebras.
2. The G_2-associative algebras satisfy the condition
$$\mu(x, \mu(y, z)) - \mu(y, \mu(x, z)) = \mu(\mu(x, y), z) - \mu(\mu(y, x), z) \qquad (6.0.5)$$
They are called Vinberg algebras or left symmetric algebras.
3. The G_3-associative algebras satisfy the condition
$$\mu(x, \mu(y, z)) - \mu(x, \mu(z, y)) = \mu(\mu(x, y), z) - \mu(\mu(x, z), y) \qquad (6.0.6)$$
They are called pre-Lie algebras or right symmetric algebras.
4. The G_4-associative algebras satisfy the condition
$$\mu(x, \mu(y, z)) - \mu(z, \mu(y, x)) = \mu(\mu(x, y), z) - \mu(\mu(z, y), x) \qquad (6.0.7)$$
5. The G_5-associative algebras satisfy the condition
$$\mu(x, \mu(y, z)) + \mu(y, \mu(z, x) + \mu(z, \mu(x, y)) =$$
$$\mu(\mu(x, y), z) + \mu(\mu(y, z), x) + \mu(\mu(z, x), y)$$
If the product μ is skewsymmetric then the previous condition is exactly the Jacobi identity.
6. The G_6-associative algebras are the Lie-admissible algebras.

A Hom-Vinberg algebra is a bilinear map on V and a homomorphism α satisfying

$$\mu(\alpha(x), \mu(y, z)) - \mu(\alpha(y), \mu(x, z)) = \mu(\mu(x, y), \alpha(z)) - \mu(\mu(y, x), \alpha(z)) \qquad (6.0.8)$$



7 Enveloping algebras

Let $\mathbb{K}$ be a commutative ring and $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. Recall that a (left) $\mathfrak{g}$-representation of $\mathfrak{g}$ is a $\mathbb{K}$-module $\mathcal{M}$ and a $\mathbb{K}$-homomorphism

$$\begin{aligned} \mathfrak{g} \otimes \mathcal{M} &\mapsto \mathcal{M} \\ x \otimes a &\mapsto xa \end{aligned} \tag{7.0.1}$$

such that $x(ya) - y(xa) = [x, y]a$. To each Lie algebra $\mathfrak{g}$, we associate an associative $\mathbb{K}$-algebra $\mathcal{U}\mathfrak{g}$ such that every (left) $\mathfrak{g}$-representation may be viewed as (left) $\mathcal{U}\mathfrak{g}$-representation and vice-versa. The algebra $\mathcal{U}\mathfrak{g}$ is constructed as follows :

Let $T\mathfrak{g}$ be a tensor algebra of $\mathbb{K}$-module $\mathfrak{g}$,

$$T\mathfrak{g} = T^0 \oplus T^1 \oplus \cdots \oplus T^n \oplus \cdots$$

where $T^n = \mathfrak{g} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ (n times).

In particular $T^0 = \mathbb{K}1$ and $T^1 = \mathfrak{g}$. The multiplication in $T\mathfrak{g}$ is the tensor product.

$$T\mathfrak{g} = (\oplus_{n \geq 0} T^n \mathfrak{g}, \otimes)$$

Every $\mathbb{K}$-linear map $\mathfrak{g} \otimes \mathcal{M} \to \mathcal{M}$ has a unique extension to a map $T\mathfrak{g} \otimes \mathcal{M} \to \mathcal{M}$. The $\mathfrak{g}$-module is a $\mathfrak{g}$-representation if and only if the elements of $T\mathfrak{g}$ of the form $x \otimes y - y \otimes x - [x, y]$ where $x, y \in \mathfrak{g}$ annihilate $\mathcal{M}$. Consequently, we are led to introduce the two-sided ideal I generated by the elements $x \otimes y - y \otimes x - [x, y]$ where $x, y \in \mathfrak{g}$. We define the enveloping algebra $\mathcal{U}\mathfrak{g}$ of $\mathfrak{g}$ as

$$\mathcal{U}\mathfrak{g} = T\mathfrak{g}/I.$$

Thus, $\mathfrak{g}$-representations and the $\mathcal{U}\mathfrak{g}$-modules may be identified. Recall that every bimodule $\mathcal{M}$ is a $\mathfrak{g}$-module by $(x, m) \to xm - mx$, denoted by $\mathcal{M}_a$.

Let denote by σ the map

$$\mathfrak{g} \to T\mathfrak{g} \to \mathcal{U}\mathfrak{g}$$

We have

$$\sigma(x)\sigma(y) - \sigma(y)\sigma(x) = \sigma([x, y]) \qquad x, y \in \mathfrak{g}$$

Theorem 7.1 (Poincaré-Birkoff-Witt) *Assume that $\mathfrak{g}$ is a free Lie algebra over $\mathbb{K}$. Let $\{x_i\}_{i \in \gamma}$ be a fixed $\mathbb{K}$-basis of $\mathfrak{g}$ and $I = (\alpha_1, ..., \alpha_p)$ be a finite sequence of indices. Setting*

$$Y_\alpha = \sigma(X_\alpha),\ Y_I = Y_{\alpha_1} .... Y_{\alpha_p}\ ,\ Y_\emptyset = 1,$$

Then the elements $\{Y_I\}$ corresponding to a finite increasing sequences I form a $\mathbb{K}$- basis of $\mathcal{U}\mathfrak{g}$ and the map $\sigma : \mathfrak{g} \to \mathcal{U}\mathfrak{g}$ is injective.

Remark 7.1 *1. The Poincaré-Birkhoff-Witt Theorem insures that the enveloping algebra $\mathcal{U}\mathfrak{g}$ is generated by the elements y_I corresponding to the increasing sequences I.*

2. We denote by $\mathcal{S}V$ the symmetric algebra over a $\mathbb{K}$-module V. If $\mathbb{Q} \in \mathbb{K}$, then there exists a canonical bijection between $\mathcal{S}\mathfrak{g}$ and $\mathcal{U}\mathfrak{g}$ which is a $\mathfrak{g}$-module isomorphism between $\mathcal{S}\mathfrak{g}$ and $\mathcal{U}\mathfrak{g}_a$ ([17, pp.78-79])

3. Let $\mathcal{U}_{-1}\mathfrak{g} = 0$ and, for $n \in \mathbb{N}$, $\mathcal{U}_n\mathfrak{g}$ be the canonical image in $\mathcal{U}\mathfrak{g}$ of $T^0 + T^1 + ... + T^n$. The sequence $(\mathcal{U}_n\mathfrak{g})_{n\geq 0}$ increases and its union is $\mathcal{U}\mathfrak{g}$. T defines a filtration, called canonical filtration of $\mathcal{U}\mathfrak{g}$. In particular, we have $\mathcal{U}_0\mathfrak{g} = \mathbb{K}$, $\mathcal{U}_1\mathfrak{g} = \mathbb{K} \oplus \mathfrak{g}$ and $\mathcal{U}_n\mathfrak{g} \cdot \mathcal{U}_p\mathfrak{g} \subset \mathcal{U}_{n+p}\mathfrak{g}$.

If $\mathbf{G}$ denote the associate graded algebra of $\mathcal{U}\mathfrak{g}$ Then

$$\mathbf{G} = \oplus_{n\geq 0}\mathbf{G}^n$$

with $\mathbf{G}^n = \mathcal{U}_n\mathbf{g}/\mathcal{U}_{n-1}\mathbf{g}$ pour $n \geq 0$. We have $\mathbf{G}^0 = \mathbb{K}$ and $\mathbf{G}^1 = \mathfrak{g}$.

The graded algebra $\mathbf{G}$ is commutative and may be identified with the symmetric algebra $\mathcal{S}(\mathfrak{g})$ of $\mathfrak{g}$.

8 Poisson structures and quantization

The quantization by deformation was introduced by Bayen, Flato, Lichnerowitz and Sternheimer in 1977 ([8]). They aim to interpret the quantum mechanics as a deformation of the classical mechanics, the Lorentz group being a deformation of the Galileo group. Or conversely the Galileo group is a contraction (degeneration) of the Lorentz group of the quantum mechanics.

8.1 Poisson algebras

The Poisson structure emerged, in deformation theory, when we study the deformation of commutative algebra in a noncommutative algebra. It is also extremely important in many other context in representation theory.

Definition 8.1 *A Poisson algebra is a an associative commutative algebra $\mathcal{A}$ over $\mathbb{K}$ with a bilinear map $\{,\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfying $\forall f, g, h \in \mathcal{A}$*

(1) $\{f, g\} = -\{g, f\}$ *(skewsymmetry)*

(2) $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ *(Jacobi relation)*

(3) $\{h, fg\} = \{h, f\}g + f\{h, g\}$ *(Leibniz relation)*

We denote such a Poisson algebra by $(\mathbf{A}, \cdot, \{,\})$.

The Leibniz relation can be restated as : for each $f \in \mathcal{A}$ the map

$$X_f := \{f, -\} : \mathcal{A} \to \mathcal{A}$$

$$g \to \{f, g\}$$

is a derivation, called the hamiltonian vector field of f on $\mathcal{A}$. The map

$$X : \mathcal{A} \to Der_{\mathbb{K}}\mathcal{A}$$

$$f \to X_f$$

where $Der_{\mathbb{K}}\mathcal{A}$ denotes the Lie algebras of all $\mathbb{K}$-linear derivations on $\mathcal{A}$, is a homomorphism of Lie algebras.

1. A manifold M is called a *Poisson manifold* if the algebra of functions, $\mathbf{C}^\infty(M)$, has a Poisson structure.

2. A Poisson structure is determined by a skew-symmetric bilinear form on T^*M. In other word there exists a tensor field $P \in \Gamma(M, \Lambda^2 TM)$ (with TM the fibre bundle of M) such that $\{f, g\} = P(df, dg) = \sum_{i,j} P^{ij}\partial_i f \partial_j g$, where ∂_i denotes the partial derivative with respect to the local coordinate x_i. The tensor field P is called the *Poisson bivector* of $(M, \{\ ,\ \})$. A Poisson structure on M is given by a bivector $P \in \Gamma(M, \Lambda^2 TM)$ satisfying

$$\sum_h P^{ih}\partial_h P^{jk} + P^{jh}\partial_h P^{ki} + P^{kh}\partial_h P^{ij} = 0$$

3. Let $T^i_{poly} = \Gamma(M, \Lambda^i TM)$ be the space of all skew symmetric tensor fields of rank i on a manifold M, $T^0_{poly} = C^\infty(M)$ and $T_{poly} = (\oplus_{n \geq 0} T^i_{poly}, \wedge)$ the algebra of multivectors on M.

   A bivecteur $P \in \mathbf{T}^2_{poly}$ defined a Poisson structure if and only if the Schouten-Nijenhuis bracket $[P, P]_s = 0$. The operator $\delta_P := [P, -]_s$ determines the so-called *Poisson cohomology*.

Example 8.1.1 *1. Moyal product*

2. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $and g^*$ its algebraic dual. The symmetric algebra $\mathcal{S}\mathfrak{g}$ is identified to the algebra of polynomial functions on $\mathfrak{g}^*$. The Lie algebra structure of $\mathfrak{g}$ induces a linear Poisson structure on $\mathfrak{g}^*$. $\{f, g\}(x) = x([df(x), dg(x)])$ with $f, g \in \mathcal{S}\mathfrak{g}$ and $x \in \mathfrak{g}^*$. Let $(e_i)_{i=1\ldots n}$ be a basis of $\mathfrak{g}$, $(e^i)_{i=1\ldots n}$ the dual basis and $x = \sum_{i=1}^n x_i e^i \in \mathfrak{g}^*$, $\{f, g\} = P_0(df, dg)$ with P_0 the bivector defined by*

$$P_0 = \frac{1}{2}\sum_{i,j} P_0^{ij}\partial_i \wedge \partial_j \quad \text{where} \quad P_0^{ij}(x) = \sum_k C_{ij}^k x_k \qquad (8.1.1)$$

where C_{ij}^k are the structure constants of $\mathfrak{g}$. Therefore, the Poisson algebra structure on $\mathcal{S}(\mathfrak{g})$.

Remark 8.1 *The Poisson bracket plays an important role in the study of integrable systems. The integrable system first appeared as mechanical system for which the equation of motion are solvable by quadrature (sequence of operation which included only algebraic operation, integration and application of the inverse function theorem). The first main result due to Liouville applying Hamilton's result was that the mechanical system with n degrees of freedom of the form*

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} i = 1, \ldots, n$$

If a function H in the coordinate p_i, q_i has n independent functions in involution, one of which is H, then it can be solved by quadrature. Recall that two functions f and g are in involution if their Poisson bracket

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

vanishes and f is called the first integral of the system if f and H are in involution.

Remark 8.2 *In the theory of Quantization deformation ([8]), one associates to the Poisson structure a formal deformation of the associative commutative algebra $\mathcal{C}^\infty(M)$, called star product, see chapter 8. The existence of a star product for every Poisson structure was established by Kontsewich see [41].*

8.2 Poisson-Hopf algebras

Let $H = (V, \mu, \eta, \Delta, \varepsilon)$ be a commutative Hopf algebra and the bracket $\{\cdot, \cdot\}$ is a Poisson structure on H with its associative structure. The Hopf algebra H is said Hopf-Poisson algebra if the comultiplication $\Delta : H \to H \otimes H$ is a Poisson algebra homomorphism. The Poisson bracket on $H \otimes H$ being defined by

$$\{a \otimes b, c \otimes d\} = \mu(a \otimes c) \otimes \{b, d\} + \{a, c\} \otimes \mu(b \otimes d)$$

That is

$$\Delta(\{x, y\}) = \{\Delta(x), \Delta(y)\}$$

8.3 Star-product

Let $\mathcal{A} = \mathcal{C}^\infty(M)$ be the commutative associative algebra of functions $\mathcal{C}^\infty$ à with real or complex values on a Poisson manifold M with the Poisson structure P. The Poisson bracket on $\mathcal{A}$ is denoted by $\{\ ,\ \}$.

Definition 8.2 *A $*$-product on $\mathcal{A}$ is an associative formal deformation of $\mathcal{A}$*

$$f *_\lambda g = \sum_{r=0}^{\infty} \lambda^r B_r(f,g)$$

such that :

1. *The $*$-product imbedded in $\mathcal{A}[[\lambda]]$ of formal series in λ with coefficients in $\mathcal{A}$ is associative, ie :*

$$\forall r \in N, \quad \sum_{i=0}^{r} (B_i(B_{r-i}(f,g),h) - (B_i(f, B_{r-i}(g,h))) = 0.$$

2. $B_0(f,g) = fg$ *(pointwise product).*
3. $B_1(f,g) - B_1(g,f) = \{f,g\}$ *et*$B_r(f,g) = (-1)^r B_r(g,f)$.
4. $B_r(f,1) = B_r(1,f) = 0 \quad \forall r > 0$
5. B_r *are bilinear.*

- The condition 3 implies that $[f,g] := \frac{1}{2\lambda}(f *_\lambda g - g *_\lambda f)$ is the deformation of the Lie structure $\{\ ,\ \}$.
- The condition $B_1(f,g) - B_1(g,f) = \{f,g\}$ ensures the "correspondence principle" between classical and quantum mechanics, that

$$\frac{f *_\lambda g - g *_\lambda f}{\lambda}|_{\lambda=0} = \{f,g\}$$

8.4 Existence of star-product : Kontsevich theorem

First, the existence of star-product has been proved in the case of symplectic manifold by M. De Wilde and P.B.A. Lecomte [15] and by B.V. Fedosov [19].

Let (M^{2n}, ω) be a symplectic manifold.

Consider a deformation quantization of $\mathcal{A} = \mathcal{C}^\infty(M)$, this is a formal deformation of the commutative multiplication on $\mathcal{A}$ to a noncommutative $*$-product $f * g = fg + \frac{\lambda}{2}\{f,g\} + \lambda^2 B_2(f,g) + \cdots$ where B_i are bidifferential operators.

These deformations were classified by Fedosov [19] up to natural equivalence. They are in one to one correspondence with their characteristic classes, formal series $\Omega = \Omega_0 + \lambda\Omega_1 + \cdots$ of cohomology classes $\Omega_i \in H^2(M, \mathbb{C})$, such that $-\Omega_0$ is the class of the symplectic form.

Such a $*$-product extends to an associative multiplication on $\mathcal{A}^\lambda := \mathcal{C}^\infty(M)[[\lambda]]$ making this into an algebra over $\mathbb{C}[[\lambda]]$. Locally, this algebra is given by the Weyl quantization of $\mathcal{C}^\infty(\mathbb{R}^{2n})$. The Weyl algebra $\mathcal{A}_{2n}$ is the algebra of formal power series $\mathbb{C}[[y_1, \cdots, y_{2n}]]$ with coefficients in $\mathbb{C}[[\lambda]]$ and Moyal product.

The polynomial Weyl algebra $\mathcal{A}_{2n}^{pol}$ over the ring $\kappa = \mathbb{C}[t, t^{-1}]$ is the space of polynomials $\kappa[p_1, \cdots, p_n, q_1, \cdots, q_n]$ with the Moyal product associated with the bivector

$$\alpha = \frac{1}{2}\sum_{i=1}^{n}(\frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial q_i} - \frac{\partial}{\partial q_i} \otimes \frac{\partial}{\partial p_i}) \in End_{\mathbb{C}(\mathcal{A}_{2n}^{pol} \otimes \mathcal{A}_{2n}^{pol})}$$

The Moyal product is given by the formula

$$f * g = m(exp(t\alpha)(f \otimes g))$$

where $m(f \otimes g) = fg$ is the standard commutative product on polynomials.

This algebra is generated by p_i, q_i with defining relations $p_i * q_j - q_j * p_i = t\delta_{ij}$. (The physicists use $\hbar = \imath t$). It is isomorphic via $q_i \to x_i, p_i \to \frac{\partial}{\partial x_i}$ to the algebra of differential operators in n variables with coefficients in $\kappa[x_1, \cdots, x_n]$.

The Weyl algebra is $\mathbb{Z}$-graded for the assignment

$$deg\, q_i = deg\, p_i = 1, \quad deg\, t = 2$$

We also consider the completed Weyl algebra $\mathcal{A}_{2n}$ over the formal Laurent series $\mathbb{C}((t))$. As a vector space $\mathcal{A}_{2n} = \mathbb{C}[[p_i, q_i]]((t))$. The product is defined by the same Moyal formula.

The completed Weyl algebra contains the subring $\mathcal{A}_{2n}^+ = \mathbb{C}[[p_i, q_i, t]]$, an algebra over $\mathbb{C}[[t]]$. The filtration of $\mathcal{A}_{2n}^+$ by degree defines a topology on $\mathcal{A}_{2n}^+, \mathcal{A}_{2n}$.

The algebras $\mathcal{A}_{2n}, \mathcal{A}_{2n}^+$ are sometimes viewed as a Lie algebras with the bracket

$$[f, g]_\lambda = \frac{1}{\lambda}(f * g - g * f)$$

The map $f \to [f, \;]_\lambda$ from $\mathcal{A}_{2n}^{pol}$ to the Lie algebra $Der(\mathcal{A}_{2n}^{pol})$ of derivations of the Weyl algebra defines an exact sequence of Lie algebra homomorphisms

$$0 \to \kappa \to \mathcal{A}_{2n}^{pol} \to Der(\mathcal{A}_{2n}^{pol}) \to 0$$

It is known from [20] that the Hochschild homology of $\mathcal{A}_{2n}^{pol}$, the polynomial Weyl algebra, is zero except in degree $2n$. Thus, the Hochschild cohomology with the coefficients in its dual is one dimensional and concentrated in degree $2n$.

An important property of the deformations given by $*$-product is that they admit a trace, i.e a linear functional τ on compactly supported functions with values in $\mathbb{C}((\epsilon))$ such that $\tau(f * g) = \tau(g * f)$. On the Weyl algebra, there is a canonical trace given by the integral with respect to the Liouville measure.

Theorem 8.1 (Kontsevish) *Every complex Poisson structure admits a star product.*

9 Pre-Lie algebras

Pre-Lie algebras are sometimes called *Vinberg algebras*, as they appear in the work of E. B. [?] under the name "left-symmetric algebras" on the classification of homogeneous cones. They appear independently at the same time in the work of M. Gerstenhaber on Hochschild cohomology and deformations of algebras, under the name "pre-Lie algebras" which is now the standard terminology. The notion itself can be however traced back to the work of A. Cayley which, in modern language, describes *the* pre-Lie algebra morphism F_a from the pre-Lie algebra of rooted trees into the pre-Lie algebra of vector fields on $\mathbb{R}^n$ sending the one-vertex tree to a given vector field a.

9.1 Definition and general properties

A *left pre-Lie algebra* over a field k is a k-vector space A with a bilinear binary composition $\rhd$ that satisfies the left pre-Lie identity:

$$a \rhd (b \rhd c) - (a \rhd b) \rhd c = b \rhd (a \rhd c) - (b \rhd a) \rhd c, \tag{9.1.1}$$

for $a, b, c \in A$. Analogously, a *right pre-Lie algebra* is a k-vector space A with a binary composition $\lhd$ that satisfies the right pre-Lie identity:

$$(a \lhd b) \lhd c - a \lhd (b \lhd c) = (a \lhd c) \lhd b - a \lhd (c \lhd b). \tag{9.1.2}$$

The left pre-Lie identity rewrites as:

$$L_{[a,b]} = [L_a, L_b], \tag{9.1.3}$$

where $L_a : A \to A$ is defined by $L_a b = a \rhd b$, and where the bracket on the left-hand side is defined by $[a, b] := a \rhd b - b \rhd a$. As an easy consequence this bracket satisfies the Jacobi identity: If A is unital (i.e. there exists $1_A \in A$ such that $1_A \rhd a = a \rhd 1_A = 1_A$ for any $a \in A$) it is immediate thanks to the fact that $L : A \to EndA$ is injective. If not, we can add a unit by considering $\overline{A} := A \oplus k.1_A$ and extend $L : \overline{A} \to End\overline{A}$ accordingly. As any right pre-Lie algebra $(A, \lhd)$ is also a left pre-Lie algebra with product $a \rhd b := b \lhd a$, one can stick to left pre-Lie algebras, what we shall do unless specifically indicated.

9.2 Pre-Lie algebras of vector fields

9.2.1 Flat torsion-free connections

Let M be a differentiable manifold, and let ∇ the covariant derivation operator associated to a connection on the tangent bundle TM. The covariant derivation is a bilinear operator on vector fields (i.e. two sections of the tangent bundle): $(X,Y) \mapsto \nabla_X Y$ such that the following axioms are fulfilled:

$$\nabla_{fX} Y = f\nabla_X Y,$$
$$\nabla_X(fY) = f\nabla_X Y + (X.f)Y \text{ (Leibniz rule).}$$

The torsion of the connection τ is defined by:

$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{9.2.1}$$

and the curvature tensor is defined by:

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. \tag{9.2.2}$$

The connection is *flat* if the curvature R vanishes identically, and *torsion-free* if $\tau = 0$. The following crucial observation by Y. Matsushima, as early as 1968 is an immediate consequence of (9.1.3):

Proposition 9.1 *For any smooth manifold M endowed with a flat torsion-free connection ∇, the space $\chi(M)$ of vector fields is a left pre-Lie algebra, with pre-Lie product given by:*

$$X \rhd Y := \nabla_X Y. \tag{9.2.3}$$

Note that on $M = \mathbb{R}^n$ endowed with its canonical flat torsion-free connection, the pre-Lie product is given by:

$$(f_i\partial_i) \rhd (g_j\partial_j) = f_i(\partial_i g_j)\partial_j. \tag{9.2.4}$$

10 Other related algebraic structures

10.1 NAP algebras

NAP algebras (NAP for Non-Associative Permutative) appear under this name in Loday, and also under the name “left- (right-)commutative algebras”. They can be seen in some sense as a “simplified version” of pre-Lie algebras.

A *left NAP algebra* over a field k is a k-vector space A with a bilinear binary composition $\blacktriangleright$ that satisfies the left NAP identity:

$$a \blacktriangleright (b \blacktriangleright c) = b \blacktriangleright (a \blacktriangleright c). \tag{10.1.1}$$

for any $a, b, c \in A$. Analogously, a *right NAP algebra* is a k-vector space A with a binary composition $\blacktriangleleft$ satisfying the right NAP identity:

$$(a \blacktriangleleft b) \blacktriangleleft c = (a \blacktriangleleft c) \blacktriangleleft b. \tag{10.1.2}$$

As any right NAP algebra is also a left NAP algebra with product $a \blacktriangleright b := b \blacktriangleleft a$, one can stick to left NAP algebras, what we shall do unless specifically indicated.

10.2 Novikov algebras

A Novikov algebra is a right pre-Lie algebra which is also left NAP, namely a vector space A together with a bilinear product $*$ such that for any $a, b, c \in A$ we have:

$$a * (b * c) - (a * b) * c = a * (c * b) - (a * c) * b, \tag{10.2.1}$$
$$a * (b * c) = b * (a * c). \tag{10.2.2}$$

Novikov algebras first appeared in hydrodynamical equations ([**?**, **?**]). The prototype is a commutative associative algebra together with a derivation D, the Novikov product being given by:

$$a * b := (Da)b. \tag{10.2.3}$$

The free Novikov algebra on a set of generators has been given in [**?**, Section 7] in terms of some classes of rooted trees.

10.3 Assosymmetric algebras

An assosymmetric algebra is a vector space endowed with a bilinear operation which is both left and right pre-Lie, which means that the associator $a * (b * c) - (a * b) * c$ is symmetric under the permutation group S_3. This notion has been introduced by E. Kleinfeld as early as 1957. All associative algebras are obviously assosymmetric, but the converse is not true.

10.4 Dendriform algebras

A *dendriform algebra* (Loday) over the field k is a k-vector space A endowed with two bilinear operations, denoted $\prec$ and $\succ$ and called right and left products, respectively, subject to the three axioms below:

$$(a \prec b) \prec c = a \prec (b \prec c + b \succ c) \tag{10.4.1}$$
$$(a \succ b) \prec c = a \succ (b \prec c) \tag{10.4.2}$$

$$a \succ (b \succ c) \quad = \quad (a \prec b + a \succ b) \succ c. \tag{10.4.3}$$

One readily verifies that these relations yield associativity for the product

$$a * b := a \prec b + a \succ b. \tag{10.4.4}$$

However, at the same time-orderingthe dendriform relations imply that the bilinear product $\rhd$ defined by:

$$a \rhd b := a \succ b - b \prec a, \tag{10.4.5}$$

is left pre-Lie. The associative operation $*$ and the pre-Lie operation $\rhd$ define the same Lie bracket, and this is of course still true for the opposite (right) pre-Lie product $\lhd$:

$$[\![a, b]\!] := a * b - b * a = a \rhd b - b \rhd a = a \lhd b - b \lhd a.$$

In the commutative case (commutative dendriform algebras are also named *Zinbiel algebras* (Loday), the left and right operations are further required to identify, so that $a \succ b = b \prec a$. In this case both pre-Lie products vanish.

10.5 Post-Lie algebras

Post-Lie algebras have been introduced by B. Vallette [?] independently to the introduction of the closely related notion of D-algebra in [?]. A left post-Lie algebra on a field k is a k-vector space A together with a bilinear binary product $\rhd$ and a Lie bracket $[-,-]$ such that

$$\begin{aligned} a \rhd [b, c] &= [a \rhd b, c] + [b, a \rhd c], \\ [a, b] \rhd c &= a \rhd (b \rhd c) - (a \rhd b) \rhd c - b \rhd (a \rhd c) + (b \rhd a) \rhd c \end{aligned}$$

for any $a, b, c \in A$. In particular, a post-Lie algebra A is a pre-Lie algebra if and only if the Lie bracket vanishes. The space of vector fields on a Lie group is a post-Lie algebra, and the free post-Lie algebra with one generator is the free Lie algebra on the linear span of *planar* rooted trees. The binary product $\rhd$ is given by *left grafting*, which is not pre-Lie anymore because of planarity.

11 Cohomology

The deformation theories are intimately linked to cohomology. We present herein the concepts of cohomology theories for algebraic structures, and Hochschild cohomology for associative algebras.

11.1 Exact sequences and Complexes

Definition 11.1 *A chain complex $\mathcal{C}_{.}$ is a sequence of abelian groups and homomorphisms*

$$\cdots \xrightarrow{d_{n+1}} \mathcal{C}_n \xrightarrow{d_n} \mathcal{C}_{n+1} \xrightarrow{d_{n-1}} \cdots$$

with the property $d_n \circ d_{n-1} = 0$ for all n.

The homomorphisms d^n are called coboundary operators or codifferentials.

A cochain complex $\mathcal{C}^{\cdot}$ is a sequence of abelian groups and homomorphisms

$$\cdots \xrightarrow{d^{n-1}} \mathcal{C}^n \xrightarrow{d^n} \mathcal{C}^{n+1} \xrightarrow{d^{n+1}} \cdots$$

with the property $d^n \circ d^{n+1} = 0$ for all n.

A chain complex can be considered as a cochain complex by reversing the enumeration $\mathcal{C}^n = \mathcal{C}_{-n}, d^n = d_{-n}$

A complex of $\mathcal{A}$-modules is a complex for which $\mathcal{C}_n$ (respectively $\mathcal{C}^n$) are modules over a ring $\mathcal{A}$ and d_n (resp. d^n) are homomorphisms of modules.

11.2 Homology and Cohomology

Let $Im\, d_{n+1}$ (resp. $Im\, d^{n-1}$) be the image of $\mathcal{C}_{n-1}$ (resp. $\mathcal{C}^{n-1}$) by d_{n-1} (resp.d^{n-1}) and $Kerd_n$ (resp. $Kerd^n$) be the kernel of d_n (resp. d^n). Since $d_n \circ d_{n+1} = 0$, we have $Im\, d_{n+1} \subset Kerd_n$.

A homology of a chain complex $\mathcal{C}$ is the group $H_n(\mathcal{C}) = Kerd_n/Imd_{n+1}$.

A cohomology of a chain complex $\mathcal{C}$ is the group $H^n(\mathcal{C}) = Kerd^n/Imd^{n-1}$.

The elements of $\mathcal{C}_n$ are called n-dimensional chains, the elements of $\mathcal{C}^n$ are n-dimensional cochains, the elements of $Z^n := Kerd^n$ (resp. $Z_n := Kerd_n$) are n-dimensional cocycles (resp. cycles), the elements of $B^n := Imd^{n-1}$ (resp. $B_n := Imd_{n+1}$) are n-dimensional coboundaries (resp. boundaries).

If $\mathcal{C}^{\cdot}$ is a complex of $\mathcal{A}$-modules, its cohomology is an $\mathcal{A}$-module. A complex is said to be acyclic (or an exact sequence) if $H^n(\mathcal{C}) = 0$ for all n.

A morphism $f : \mathcal{C}^{\cdot} \to \mathcal{C}^{\cdot}$ is a family of group (module) homomorphisms $f^n : \mathcal{C}^n \to D^n$ commuting with differentials, that is $f^{n+1} \circ d^n_{\mathcal{C}} = d^n_{\mathcal{C}} \circ f^n$. A morphism f induces a morphism of cohomology $H^{\cdot}(f) = \{H^n(f) : H^n(\mathcal{C}^{\cdot}) \to H^n(\mathcal{D}^{\cdot})\}$ by the formula $\{$the class of cocycle $c\} = \{$the class of cocycle $f(c)\}$

11.3 Hochschild Cohomology

We recall here the definitions of the Hochschild cohomological groups of associative algebra.

Let $\mathcal{A}$ be an associative algebra on the $\mathbb{K}$-vector space V defined by the multiplication μ. A Hochschild p-cochain is a linear map from $V^{\otimes p}$ to V. We denote by $\mathcal{C}^p(\mathcal{A},\mathcal{A})$ the group of all p-cochains. The boundary map :

$$\delta^p : \begin{array}{c} \mathcal{C}^p(\mathcal{A},\mathcal{A}) \to C^{p+1}(\mathcal{A},\mathcal{A}) \\ \varphi \to \delta^p\varphi \end{array}$$

is defined by : for $(x_1,...,x_{p+1}) \in V^{\otimes(p+1)}$

$$\begin{aligned}\delta^p\varphi\left(x_1,...,x_{p+1}\right) &= \mu\left(x_1,\varphi\left(x_2,...,x_{p+1}\right)\right) \\ &+\sum_{i=1}^{p}\left(-1\right)^i\varphi\left(x_1,...,\mu\left(x_i,x_{i+1}\right),...,x_p\right)+\left(-1\right)^{p+1}\mu\left(\varphi\left(x_1,...,x_p\right),x_{p+1}\right)\end{aligned}$$

The kernel of δ^p in $C^p(\mathcal{A},\mathcal{A})$ is the group of p-cocycles :

$$Z^p\left(\mathcal{A},\mathcal{A}\right)=\left\{\varphi\ :\ V^{\otimes p}\to V\ /\ \delta^p\varphi=0\right\}$$

And the image of δ^{p-1} in $\Omega^p\left(V\right)$ is the group of p-coboundaries :

$$B^p\left(\mathcal{A},\mathcal{A}\right)=\left\{\varphi\ :\ V^{\otimes p}\to V\ /\ \varphi=\delta^{p-1}f,\ f\in C^{p-1}(\mathcal{A},\mathcal{A})\right\}$$

The Hochschild cohomology group of the algebra $\mathcal{A}$ with coefficient in itself is

$$H^p\left(\mathcal{A},\mathcal{A}\right)=Z^p\left(\mathcal{A},\mathcal{A}\right)/B^p\left(\mathcal{A},\mathcal{A}\right).$$

In the particular cases,

- $H^0\left(\mathcal{A},\mathcal{A}\right)$ gives the center of the algebra.

  $$Z(\mathcal{A})=H^0\left(\mathcal{A},\mathcal{A}\right)=\{x\in\mathcal{A}\ :\mu\left(x,y\right)=\mu\left(y,x\right)\forall y\in V)$$

  .

- $H^1\left(\mathcal{A},\mathcal{A}\right)$ gives the algebra of derivations :

  $$Der(\mathcal{A}))=\{f:V\to V,\ \forall x,y\in V\quad f\left(\mu\left(x,y\right)\right)=\mu\left(f\left(x\right),y\right)+\mu\left(x,f\left(y\right)\right)\}$$

  modul internal derivations, i.e; derivation of the form $f_a(x)=xa-ax$ for $x\in\mathcal{A}$ and some fixed $a\in\mathcal{A}$.

Remark 11.1 *In the general case, where the cohomology is with values in a bi-module M,*

$$H^1(\mathcal{A}, M) = Der(A, M)/IDer(\mathcal{A}, M)$$

where $IDer(\mathcal{A}, M)$ denotes the subspace of internal derivations i.e. derivations of the form $f_m(a) = am - ma$ for $a \in \mathcal{A}$ and some fixed $m \in M$.

The following cohomology groups play an important in deformations theories.

- $B^2(\mathcal{A}, \mathcal{A}) = \{\varphi : V \otimes V \to V \ / \ \varphi = \delta_1 f, \ f \in End(V)\}$
  $\forall x, y \in V \qquad \delta^1 f(x, y) = \mu(f(x), y) + \mu(x, f(y)) - f(\mu(x, y))$
- $Z^2(\mathcal{A}, \mathcal{A}) = \{\varphi \in \Omega^2(V) \ / \ \delta^2\varphi = 0\}$
  $\forall x, y, z \in V \qquad \delta^2\varphi(x, y, z) = \mu(\varphi(x, y), z) - \mu(x, \varphi(y, z)) +$
  $\varphi(\mu(x, y), z) - \varphi(x, \mu(y, z))$.
- $B^3(\mathcal{A}, \mathcal{A}) = \{\Psi : V \otimes V \otimes V \to V \ / \ \Psi = \delta^2\varphi$ where $\varphi \in \Omega^2(V)\}$
- $Z^3(\mathcal{A}, \mathcal{A}) = \{\varphi \in \Omega^3(V) \ / \ \delta^3\varphi = 0\}$
  $\forall x, y, z, t \in V \qquad \delta^3\varphi(x, y, z, t) = \mu(x, \varphi(y, z, t)) - \varphi(\mu(x, y), z, t) +$
  $\varphi(x, \mu(y, z), t) - \varphi(x, y, \mu(z, t)) + \mu(\varphi(x, y, z), t)$.

Remark 11.2 *1. $B^n(\mathcal{A}, \mathcal{A}) \subset Z^n(\mathcal{A}, \mathcal{A})$ because $\delta^n \circ \delta^{n-1} = 0$.*

2. *If V is finite-dimensional, the dimension of the 2-coboundaries space corresponds to the dimension of the orbit of $\mathcal{A}$ (as a differential variety) and the dimension of the 2-cocycles space corresponds to the dimension of the Zariski tangent space to the variety at the point $\mathcal{A}$.*
3. *dim $H^2(\mathcal{A}, \mathcal{A})$ =dim $Z^2(\mathcal{A}, \mathcal{A})$-dim $B^2(A, A)$.*
4. *The dimension of these spaces are algebraic invariants which means that the dimension is the same for all isomorphic algebras. Then it permits to distinguish two algebras if they are different.*

11.4 Gerstenhaber algebra

We define two maps

$$\circ : C^d(\mathcal{A}, \mathcal{A}) \times C^e(\mathcal{A}, \mathcal{A}) \to C^{d+e-1}(\mathcal{A}, \mathcal{A})$$

by

$$(\varphi \circ \psi)(a_1, \cdots, a_{d+e-1}) = \sum_{i=0}^{d-1} (-1)^{i(e-1)} \varphi(a_1, \cdots, a_i, \psi(a_{i+1}, \cdots, a_{i+e}), \cdots)$$

And

$$[.,.]_G : C^d(\mathcal{A},\mathcal{A}) \times C^e(\mathcal{A},\mathcal{A}) \to C^{d+e-1}(\mathcal{A},\mathcal{A})$$

by

$$[\varphi,\psi]_G = \varphi \circ \psi - (-1)^{(e-1)(d-1)}\psi \circ \varphi.$$

The space $(C(\mathcal{A},\mathcal{A}),\circ)$ is a pre-Lie algebra and $(C(\mathcal{A},\mathcal{A})[.,.]_G)$ is a graded Lie algebra (Lie superalgebra). The bracket $[.,.]_G$ is called Gerstenhaber's bracket. The square of $[\mu,.]_G$ vanishes and defines the 2-coboundary operator. The multiplication μ of $\mathcal{A}$ is associative if $[\mu,\mu]_G = 0$.

12 Formal power series ring and spaces

Let $\mathbb{K}$ be an algebraically field of characteristic 0. The power series are a generalization of polynomials. Recall that it is often convenient in investigating the properties of finite sequence $(a_0, a_1, \cdots, a_n)$ in $\mathbb{K}$ to examine the polynomial $f(t) = a_0 + a_1 t + \cdots + a_n t^n$, the generating function of the sequence. Given an infinite sequence $(a_0, a_1, \cdots, a_n, \cdots)$, we can write $f(t) = a_0 + a_1 t + \cdots + a_n t^n + \cdots$. The expression is called a power series. The set of power series in one variable t over $\mathbb{K}$ is denoted by $\mathbb{K}[[t]]$

$$\mathbb{K}[[t]] = \{\sum_{i\geq 0} a_i t^i \quad a_i \in \mathbb{K}\}$$

We assume that the elements of $\mathbb{K}$ commute with the parameter t.

One consider that two power series $f(t) = a_0 + a_1 t + \cdots + a_n t^n + \cdots$ and $g(t) = b_0 + b_1 t + \cdots + b_n t^n + \cdots$ of $\mathbb{K}[[t]]$ are equal if $a_i = b_i$ for all $i \geq 0$.

The operations addition, multiplication are defined by exactly the same formulas as for polynomials.

Let $f(t) = \sum_{i\geq 0} a_i t^i$ and $g(t) = \sum_{i\geq 0} b_i t^i$ are two power series in $\mathbb{K}[[t]]$. Then sum is a power series defined by

$$f(t) + g(t) = \sum_{i\geq 0} (a_i + b_i) t^i$$

The multiplication is the power series

$$f(t)g(t) = \sum_{k\geq 0} (\sum_{i=0}^{k} a_i b_{k-i}) t^k$$

The set $\mathbb{K}[[t]]$ is an abelian group with respect to addition, the multiplication is associative and distributive over addition. In fact, $\mathbb{K}[[t]]$ is a ring with the previous operations, the identity for the multiplication being the constant power series 1.

12.1 Power series properties

In the following, we show while a power series in $\mathbb{K}[[t]]$ admits an inverse power series.

Theorem 12.1 *Any power series $f(t) = \sum_{i\geq 0} a_i t^i$ in $\mathbb{K}[[t]]$ with a nonzero free term a_0 has an inverse power series $f(t)^{-1}$.*

Proof 3 *We must find a power series $g(t) = \sum_{i\geq 0} b_i t^i$ such that $f(t)g(t) = 1$. We obtain $\sum_{k\geq 0}(\sum_{i=0}^{k} a_i b_{k-i} t^k) = 1$. This holds if $a_0 b_0 = 1$ and $\sum_{i=0}^{k} a_i b_{k-i} = 0$ for all $k > 0$. Then $b_0 = a_0^{-1}$ (which exists because $a_0 \neq 0$). The next equation $a_0 b_1 + a_1 b_0 = 0$ implies $b_1 = a_0^{-2} a_1$. Assume that we have considered the coefficients of $1, t, t^2, \cdots, t^{n-1}$ and determined $b_0, b_1, \cdots, b_{n-1}$. Equating the coefficient of t^n to 0 gives the equation*

$$a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = 0$$

and hence $b_n = -a_0^{-1}(a_1 b_{n-1} + \cdots + a_n b_0)$. Since $b_0, b_1, \cdots, b_{n-1}$ have already been determined, this gives the value for b_n.

Every rational expression $\frac{f(t)}{g(t)}$ of two power series of $\mathbb{K}[[t]]$ is a power series in $\mathbb{K}[[t]]$ if the free term b_0 of $g(t)$ is nonzero. In particular, the rational function of polynomials can be represented in the form of power series.

Remark 12.1 *The theorem may be interpreted as $\mathbb{K}[[t]]$ is a local ring whose maximal ideal is $t\mathbb{K}[[[t]]$, the ideal generated by t.*

Theorem 12.2 *Let $f(t) = \sum_{i\geq 0} a_i t^i$ be a power series in $\mathbb{K}[[t]]$ such that $a_0 \neq 0$. For any b_0 such that $b_0^k = a_0$ there exists a power series $g(t) = \sum_{i\geq 0} b_i t^i$ satisfying $(g(t))^k = f(t)$.*

Proof 4 *The existence of b_0 is guaranteed by the conditions of the theorem. hence, $b_0 \neq 0$. The equality of first degree term implies that $b_1 = \frac{1}{k} b_0^{-(k-1)} a_1$. We assume that the coefficients $b_0, b_1, \cdots, b_{n-1}$ have already been determined. We identify the coefficients of degree t^{n+1} in the equation $(g(t))^k = f(t)$.*

We set $u(t) = b_0 + b_1 t + \cdots + b_n t^n$ and $v(t) = b_{n+2} t^{n+2} + \cdots$. Then, the left-hand side in the equation has the form $(u(t) + b_{n+1} t^{n+1} + v(t))^k$. Using the binomial formula to expand this expression, we see that terms of degree $n+1$ can arise only from the terms $u(t)^k + k u(t)^{k-1} b_{n+1} t^{n+1}$. It has the form $(F(b_0, b_1, \cdots, b_n) + k b_0^{k-1} b_{n+1}) t^{n+1}$, where $F(b_0, b_1, \cdots, b_n)$ is coefficient in $u(t)$. Therefore, $b_{n+1} = b_0^{-(k-1)}(a_{n+1} - F(b_0, b_1, \cdots, b_n))$.

If $\mathbb{K} = \mathbb{R}$ and according to this theorem, if $a_0 > 0$, then the power series $f(t) = \sum_{i\geq 0} a_i t^i$ has unique kth root power series $\sqrt{f(t)}$ with a positive free term.

Let $n > 0$. We set $\mathbb{K}_n[t]$ to be the polynomials algebra obtained as a quotient of $\mathbb{K}[t]$ by the ideal generated by t^n.

There exists a morphism which associate to every power series $\sum_{i\geq 0} a_i t^i$ its class $\sum_{i\geq 0}^{n-1} a_i t^i$ modulo (t^n).

Set

$$\begin{array}{cccc} \pi_n : & \mathbb{K}[[t]] & \to & \mathbb{K}_n[t] \\ & \sum_{i\geq 0} a_i t^i & \to & \sum_{i\geq 0}^{n-1} a_i t^i \end{array}$$

The map π_n is surjective and a kernel generated by $t^n\mathbb{K}[[t]]$. It induces the following isomorphism

$$\mathbb{K}[[t]]/t^n \cong \mathbb{K}[t]/t^n.$$

Proposition 12.1 *The map $\pi_n : \mathbb{K}[[t]] \to \underleftarrow{lim}\mathbb{K}[t]/(t^n)$ is an algebra isomorphism.*

Remark 12.2 *The space $\mathbb{K}[[t]]$ is endowed with the t-adique topology. The subsets $t^n\mathbb{K}[[t]]$ form a neighborhood basis of 0. The space $\mathbb{K}[[t]]$ is Hausdorff and complete.*

12.2 Formal spaces

Let V be a $\mathbb{K}$-vector space. We denote by $V[[t]]$ the space of formal series with coefficients in V.

$$V[[t]] = \{\sum_{i\geq 0} x_i t^i \quad x_i \in V\}$$

The space $V[[t]]$ is a $\mathbb{K}[[t]]$-module with the action

$$(\sum_{i\geq 0} a_i t^i, \sum_{j\geq 0} x_j t^j) \to \sum_{p\geq 0}\sum_{i+j=p} a_i x_j t^p$$

where $a_i \in \mathbb{K}$ and $x_i \in V$.

Note that V is a submodule of $V[[t]]$. One can obtain an extension of V with a structure of vector space by extending the coefficient domain from $\mathbb{K}$ to $\mathbb{K}((t))$, the quotient power series field of $\mathbb{K}[[t]]$.

One notes that $V[[t]]$ is a projective limit of the spaces $V[[t]]_n := V[[t]]/(t^{n+1}V[[t]])$.

Let f be a $\mathbb{K}[[t]]$-morphism, $f : V\otimes\mathbb{K}[[t]] \to V[[t]]$, defined by $f(x\otimes\lambda) = \lambda x$.

Proposition 12.2

1. *f is injective.*
2. *If V is finite dimensional, then f is surjective. Hence, $V\otimes\mathbb{K}[[t]]$ and $V[[t]]$ are isomorphic.*
3. *If $(e_i)_{i\in I}$ is a basis of V then $(e_i)_{i\in I}$ is a topological basis of $V[[t]]$.*

$(e_i)_{i\in I}$ is a topological basis of $V[[t]]$ means that elements of $V[[t]]$ are of the form $\sum_{i\in I}\lambda_i e_i$ where λ_i span $\mathbb{K}[[t]]$ and tend to 0 following the filter of the complements of finite subsets of I.

Proof 5 1. *An element of $V\otimes\mathbb{K}[[t]]$ is of the form $X=\sum_{i\in J}e_i\otimes\lambda_i$ where J is a finite subset of I, $\lambda_i=\sum_q\lambda_{i,q}t^q$. Then, $f(X)=\sum_q(\sum_{i\in J}\lambda_{i,q}e_i)t^q$. Obviously, if $f(X)=0$ then $X=0$. Therefore f is injective.*

2. *An element of $V[[t]]$ is of the form $\sum_q(\sum_{i\in J_q}a_{q,i}e_i)t^q$ where J_q is finite subset of I and $a_{q,i}\in\mathbb{K}$. The previous element is in $V\otimes\mathbb{K}[[t]]$ if and only if the union of J_q is finite.*

3. *Setting $Z_i=q\in\mathbb{N}/i\in J_q$ and $\lambda_i=\sum q\in Z_i a_{q,i}t^q$*

   One may see that λ_i tends to 0 following the filter of complements of finite subsets of I. Hence, $\eta=\sum_{i\in I}\lambda_i e_i$. The converse follows from the fact that the family $(\lambda_i e_i)$ is "summable".

Lemma 12.1 *A $\mathbb{K}[[t]]$-module W is of the form $V[[t]]$ if and only if W is Hausdorff, complete and torsion free (ie the annihilator in $\mathbb{K}[[t]]$ of an nonzero element in $V[[t]]$ is trivial ($tx=0\Rightarrow x=0$)).*

13 Formal deformations

The formal deformations are the more popular approach to deform algebraic structure. They were introduced by Gerstenhaber in a series of papers for associative algebras and extended to other algebraic structures in a similar way. The materials set in this section is due to Gerstenhaber and his collaborators [29], [64] and may be found also in [39], [66].

13.1 Definitions

Let V be a $\mathbb{K}$-vector space and $\mathcal{A}_0=(V,\mu_0)$ be an associative algebra. Let $\mathbb{K}[[t]]$ be the power series ring in one variable t and coefficients in $\mathbb{K}$ and $V[[t]]$ be the set of formal power series whose coefficients are elements of V, ($V[[t]]$ is obtained by extending the coefficients domain of V from $\mathbb{K}$ to $\mathbb{K}[[t]]$). Then $V[[t]]$ is a $\mathbb{K}[[t]]$-module. When V is finite-dimensional, we have $V[[t]]=V\otimes_{\mathbb{K}}\mathbb{K}[[t]]$. One notes that V is a submodule of $V[[t]]$. Given a $\mathbb{K}$-bilinear map $f:V\times V\rightarrow V$, it admits naturally an extension to a $\mathbb{K}[[t]]$-bilinear map $f:V[[t]]\times V[[t]]\rightarrow V[[t]]$, that is, if $x=\sum_{i\geq0}a_it^i$ and $y=\sum_{j\geq0}b_jt^j$ then $f(x,y)=\sum_{i\geq0,j\geq0}t^{i+j}f(a_i,b_j)$.

Definition 13.1 *Let V be a $\mathbb{K}$-vector space and $\mathcal{A}_0 = (V, \mu_0)$ be an associative algebra. A* formal associative deformation *of $\mathcal{A}_0$ is given by the $\mathbb{K}[[t]]$-bilinear map $\mu_t : V[[t]] \times V[[t]] \rightarrow V[[t]]$ of the form*

$$\mu_t = \sum_{i \geq 0} \mu_i t^i,$$

where each μ_i is a $\mathbb{K}$-bilinear map $\mu_i : V \times V \rightarrow V$ (extended to be $\mathbb{K}[[t]]$-bilinear), such that holds for $x, y, z \in V$ the following formal associativity condition:

$$\mu_t(x, \mu_t(y, z)) - \mu_t(\mu_t(x, y), z)) = 0. \tag{13.1.1}$$

The series are infinite, then a natural question is to insure about the convergence. This is given by Artin's theorem $[\mathcal{A}]$.

Proposition 13.1 *A formal multiplication $\mu_t = \sum_{i \geq 0} \mu_i t^i$ is associative if and only if $[\mu_t, \mu_t] = 0$.*

Proof 6 *We have $[\mu_t, \mu_t](x, y, z) = 2(\mu_t(x, \mu_t(y, z)) - \mu_t(\mu_t(x, y), z))$.*

13.2 Deformation equation

The first problem is to give conditions about μ_i such that the deformation μ_t be associative, ie

$$\text{For all } x, y, z \in V \quad \mu_t\left(\mu_t\left(x, y\right), z\right) - \mu_t\left(x, \mu_t\left(y, z\right)\right) = 0$$

Expanding the left side of this equation and collecting the coefficients of t^k yields

$$\left\{ \sum_{i+j=k \quad i,j \geq 0} \mu_i\left(\mu_j\left(x, y\right), z\right) - \mu_i\left(x, \mu_j\left(y, z\right)\right) = 0 \qquad k = 0, 1, 2, \cdots \right. \tag{13.2.1}$$

This infinite system, called the *deformation equation,* gives the necessary and sufficient conditions for the associativity of μ_t. It may be written

$$\left\{ \sum_{i=0}^{k} \mu_i\left(\mu_{k-i}\left(x, y\right), z\right) - \mu_i\left(x, \mu_{k-i}\left(y, z\right)\right) = 0 \qquad k = 0, 1, 2, \cdots \right. \tag{13.2.2}$$

The first equation $(k = 0)$ is the associativity condition for μ_0.
The second shows that μ_1 must be a 2-cocycle for the Hochschild cohomology $(\mu_1 \in Z^2\left(A, A\right))$.
More generally, suppose that μ_p be the first non-zero coefficient after μ_0 in the deformation μ_t. This μ_p is called the *infinitesimal* of μ_t.

Theorem 13.1 *The map μ_p is a 2- cocycle of the Hochschild cohomology of $\mathcal{A}$ with coefficient in itself.*

Proof 7 *Take the equation in (13.2.2) with $k = p$ and $\mu_1 = \cdots = \mu_{p-1} = 0$.*

Definition 13.2 *The cocycle μ_p is said integrable if it is the first term, after μ_0, of an associative deformation.*

The integrability of μ_p implies an infinite sequence of relations which may be interpreted as the vanishing of the obstruction to the integration of μ_p.

13.3 Obstructions

Gathering the first and the last terms in the k^{th} equation of the system 13.2.2, for an arbitrary k, $k > 1$, the equation may be written

$$\delta^2\mu_k(x,y,z) = \sum_{i=1}^{k-1} \mu_i(\mu_{k-i}(x,y),z) - \mu_i(x,\mu_{k-i}(y,z))$$

Suppose that the truncated deformation $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \ldots . t^{m-1}\mu_{m-1}$ satisfies the deformation equation. The truncated deformation is extended to a deformation of order m, ie $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \ldots . t^{m-1}\mu_{m-1} + t^m\mu_m$ satisfying the deformation equation if

$$\delta^2\mu_m(x,y,z) = \sum_{i=1}^{m-1} \mu_i(\mu_{m-i}(x,y),z) - \mu_i(x,\mu_{m-i}(y,z))$$

The right-hand side of this equation is called the *obstruction* to finding μ_m extending the deformation.

By using the circle operation defined in section 11.4, we have

$$\mu_i \circ \mu_j(x,y,z) = \mu_i(\mu_j(x,y),z) - \mu_i(x,\mu_j(y,z)),$$

and the obstruction may be written

$$\sum_{i=1}^{m-1} \mu_i \circ \mu_{m-i} \text{ or } \sum_{i+j=m\ i,j\neq m} \mu_i \circ \mu_j$$

Lemma 13.1 *If f and g are bilinear maps on V, then $\delta^3(f \circ g) = \delta^2 f \circ g - f \circ \delta^2 g$*

Proof 8 *By direct calculation.*

Theorem 13.2 *The obstruction is a Hochschild 3-cocycles.*

Proof 9

$$\begin{aligned}
\delta^3\left(\sum_{i+j=m,\ i,j\neq m}\mu_i\circ\mu_j\right) &= \sum_{i+j=m\ i,j\neq m}\delta^3(\mu_i\circ\mu_j) \\
&= \sum_{i+j=m\ i,j\neq m}\delta^2\mu_i\circ\mu_j-\mu_i\circ\delta^2\mu_j \\
&= \sum_{i+j+k=m\ i,j,k\neq m}\mu_i\circ\mu_j\circ\mu_k-\mu_i\circ\mu_j\circ\mu_k=0
\end{aligned}$$

Remark 13.1

1. *The cohomology class of the element $\sum_{i+j=m,\quad i,j\neq m}\mu_i\circ\mu_j$ is the first obstruction to the integration of μ_m.*
   Suppose $m=1$ and $\mu_t=\mu_0+t\mu_1+t^2\mu_2$. The deformation equation is equivalent to
   $$\begin{cases}\mu_0\circ\mu_0=0 \quad (\mu_0 \text{ is associative})\\ \delta^2\mu_1=0 \quad (\mu_1\in Z^2(A,A))\\ \mu_1\circ\mu_1=\delta^2\mu_2\end{cases}$$
   Then $\mu_1\circ\mu_1$ is the first obstruction to integrate μ_1 and $\mu_1\circ\mu_1\in Z^3(A,A)$. The elements $\mu_1\circ\mu_1$ which are coboundaries permit to extend the deformation of order one to a deformation of order 2. But the elements of $H^3(A,A)$ gives the obstruction to the integrations of μ_1.

2. *If μ_m is integrable then the cohomological class of $\sum_{i+j=m\ i,j\neq m}\mu_i\circ\mu_j$ in $H^3(\mathcal{A},\mathcal{A})$ must be 0.*
   In the previous example μ_1 is integrable implies $\mu_1\circ\mu_1=\delta^2\mu_2$ which means that the cohomology class of $\mu_1\circ\mu_1$ vanishes.

Corollary 13.1 *If $H^3(\mathcal{A},\mathcal{A})=0$ then all obstructions vanish and every $\mu_m\in Z^2(\mathcal{A},\mathcal{A})$ is integrable.*

Hence, we have the following theorem

Theorem 13.3 *Let $\mu_t=\mu_0+t\mu_1+t^2\mu_2+....t^{m-1}\mu_{m-1}$ be $m-1$ order deformation of an algebra $(\mathcal{A}_0,\mu_0)$. Then it extends to a m order deformation if and only if the cohomology class of $\sum_{i+j=m\ i,j\neq m}\mu_i\circ\mu_j$ vanishes.*

13.4 Equivalent and trivial deformations

The next problem to consider is when two deformations are significantly different. Given two associative deformations μ_t and $\mu_t^{'}$ of μ_0, we say that they are equivalent if there is a formal isomorphism f_t which is a $\mathbb{K}\left[[t]\right]$-linear map that may be written in the form

$$f_t = Id + tf_1 + t^2 f_2 + \cdots \qquad where \quad f_i \in End_{\mathbb{K}}\left(V\right)$$

such that $\mu_t = f_t \cdot \mu_t^{'}$ defined by

$$\mu_t(x,y) = f_t^{-1} \circ \mu_t^{\prime} \circ \left(f_t\left(x\right), f_t\left(y\right)\right) \quad \text{for all } x, y \in V$$

A deformation μ_t of μ_0 is said to be *trivial* if and only if μ_t is equivalent to μ_0.

Suppose $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \cdots$ is a deformation of μ_0. They are equivalent if it exists $f_t = Id + tf_1 + t^2 f_2 + \cdots where \quad f_i \in End_{\mathbb{K}}\left(V\right)$ such that

$$f_t\left(\mu_t(x,y)\right) = \mu_t^{\prime}(f_t\left(x\right), f_t\left(y\right)) \quad \text{for all } x, y \in V \tag{13.4.1}$$

The identification of the coefficients of t gives $\mu_1 = \delta f_1$.
In general, if the deformations μ_t and $\mu_t^{\prime}$ of μ_0 are equivalent then $\mu_1^{\prime} = \mu_1 + \delta f_1$.

Therefore, we have the following proposition.

Proposition 13.2 *The integrability of μ_1 depends only on its cohomology class.*

Recall that two elements are cohomologous if there difference is a coboundary. $\delta\mu_1 = 0$ implies that $\delta\mu_1^{\prime} = \delta\left(\mu_1 + \delta f_1\right) = \delta\mu_1 + \delta\left(\delta f_1\right) = 0$.
If $\mu_1 = \delta g$ then $\mu_1^{\prime} = \delta g - \delta f_1 = \delta\left(g - f_1\right)$.

Remark 13.2 *The elements of $H^2\left(\mathcal{A}, \mathcal{A}\right)$ give the infinitesimal deformation ($\mu_t = \mu_0 + t\mu_1$).*

Proposition 13.3 *Let $\mathcal{A}_0 = (V, \mu_0)$ be an associative algebra. There is, over $\mathbb{K}[[t]]/t^2$, a one-to-one correspondence between the elements of $H^2(\mathcal{A}_0, \mathcal{A}_0)$ and the infinitesimal deformation of $\mathcal{A}_0$ defined by*

$$\mu_t(x,y) = \mu_0(x,y) + \mu_1(x,y)t, \quad \forall x, y \in V. \tag{13.4.2}$$

Proof 10 *The deformation equation is equivalent to $\delta^2\mu_1 = 0$, that is $\mu_1 \in Z^2(\mathcal{A}_0, \mathcal{A}_0)$.*

Suppose now that $\mu_t = \mu_0 + t\mu_1 + t^2\mu_2 + \cdots$ is a one parameter family of deformation of μ_0 for which $\mu_1 = \cdots = \mu_{m-1} = 0$.
The deformation equation implies $\delta\mu_m = 0$ $\left(\mu_m \in Z^2\left(A, A\right)\right)$. If further $\mu_m \in$

$B^2(A, A)$ (ie. $\mu_m = \delta g$), then setting the morphism $f_t = Id + tf_m$ we have, for all $x, y \in V$

$$\mu'_t(x, y) = f_t^{-1} \circ \mu_t \circ (f_t(x), f_t(y)) = \mu_0(x, y) + t^{m+1}\mu_{m+1}(x, y) \cdots$$

And again $\mu_{m+1} \in Z^2(\mathcal{A}, \mathcal{A})$.

Theorem 13.4 *Let μ_t be a one parameter family of deformation of μ_0. Then μ_t is equivalent to $\mu_t = \mu_0 + t^p\mu'_p + t^{p+1}\mu'_{p+1} + \cdots$ where $\mu'_p \in Z^2(\mathcal{A}, \mathcal{A})$ and $\mu'_p \notin B^2(\mathcal{A}, \mathcal{A})$.*

Corollary 13.2 *If $H^2(\mathcal{A}, \mathcal{A}) = 0$ then all deformations of $\mathcal{A}$ are equivalent to a trivial deformation.*

In fact, assume that there exists a non trivial deformation of μ_0. Following the previous theorem, this deformation is equivalent to $\mu_t = \mu_0 + t^p\mu'_p + t^{p+1}\mu'_{p+1} + \cdots$ where $\mu'_p \in Z^2(\mathcal{A}, \mathcal{A})$ and $\mu'_p \notin B^2(\mathcal{A}, \mathcal{A})$. This is impossible because $H^2(\mathcal{A}, \mathcal{A}) = 0$.

13.5 Poisson algebras and Deformations of commutative algebras

We consider now a commutative associative algebra and show that first order deformation induces a Poisson structure.

Let $\mathcal{A}_t = (V, \mu_t)$ be a deformation of a commutative associative algebra $\mathcal{A}_0 = (V, \mu_0)$. Assume that

$$\mu_t(x, y) = \mu_0(x, y) + \mu_1(x, y)t + \mu_2(x, y)t^2 + \ldots, \quad \forall x, y \in V$$

Then

$$\frac{\mu_t(x, y) - \mu_t(y, x)}{t} = \mu_1(x, y) - \mu_1(y, x) + t\sum_{i \geq 2}(\mu_i(x, y) - \mu_i(y, x))t^{i-1}.$$

Hence, if t goes to zero then $\frac{\mu_t(x,y)-\mu_t(y,x)}{t}$ goes to $\{x, y\} := \mu_1(x, y) - \mu_1(y, x)$. The previous bracket will define a structure of Poisson algebra over the commutative algebra $\mathcal{A}_0$.

In physical terms, one can regard t as a quantum parameter such as the Planck constant, the Poisson algebra $\mathcal{A}$ as the quasi classical limit of the Poisson algebra $\mathcal{A}[[t]]$ and the algebra $\mathcal{A}[[t]]$ as deformation quantization of the Poisson algebra $\mathcal{A}$.

The following lemma shows that any skewsymmetric 2-cocycle of a commutative algebra satisfies the Leibniz relation.

Lemma 13.2 *Let $\mathcal{A} = (V, \mu)$ be a commutative associative algebra and φ be a skewsymmetric 2-cochain such that $\delta^2\varphi = 0$. Then for $x, y, z \in V$*

$$\varphi(x, \mu(y, z)) = \mu(y, \varphi(x, z)) + \mu(z, \varphi(x, y)). \tag{13.5.1}$$

Proof 11 *The condition $\delta^2\varphi = 0$ is equivalent to*

$$\varphi(x, \mu(y, z)) - \varphi(\mu(x, y), z) + \mu(x, \varphi(y, z)) - \mu(\varphi(x, y), z) = 0$$

Then one has

$$\varphi(x, \mu(y, z)) = \varphi(\mu(x, y), z) - \mu(x, \varphi(y, z)) + \mu(\varphi(x, y), z) \tag{13.5.2}$$

$$\varphi(x, \mu(z, y)) = \varphi(\mu(x, z), y) - \mu(x, \varphi(z, y)) + \mu(\varphi(x, z), y) \tag{13.5.3}$$

$$\varphi(\mu(y, x), z) - \varphi(y, \mu(x, z)) = \mu(y, \varphi(x, z)) - \mu(\varphi(y, x), z). \tag{13.5.4}$$

By adding the equations 13.5.2, 13.5.3 and 13.5.4 and considering the fact that φ is skewsymmetric and μ is commutative one has

$$2\,\varphi(x, \mu(y, z)) = 2\,\mu(y, \varphi(x, z)) + 2\,\mu(z, \varphi(x, y)).$$

Lemma 13.3 *Let $\mathcal{A}_0 = (V, \mu_0)$ be a commutative associative algebra and $\mathcal{A}_t = (V, \mu_t)$ be a deformation of $\mathcal{A}_0$. Then*

$$\circlearrowleft_{x,y,z} \delta^2\mu_2(x, y, z) = \circlearrowleft_{x,y,z} \mu_2 \circ \mu_0(x, y, z),$$

where $\circlearrowleft_{x,y,z}$ stands for the cyclic summation on x, y, z.

Proof 12 *By using the commutativity of μ_0, one has*

$$\begin{aligned}
\circlearrowleft_{x,y,z} \delta^2\mu_2(x, y, z) &= \mu_2(x, \mu_0(y, z)) - \mu_2(\mu_0(x, y), z \\
&+ \mu_0(x, \mu_2(y, z)) - \mu_0(\mu_2(x, y), z) \\
&+ \mu_2(y, \mu_0(z, x)) - \mu_2(\mu_0(y, z), x \\
&+ \mu_0(y, \mu_2(z, x)) - \mu_0(\mu_2(y, z), x) \\
&+ \mu_2(z, \mu_0(x, y)) - \mu_2(\mu_0(z, x), y \\
&+ \mu_0(z, \mu_2(x, y)) - \mu_0(\mu_2(z, x), y) \\
&= \circlearrowleft_{x,y,z} \mu_2 \circ \mu_0(x, y, z).
\end{aligned}$$

Lemma 13.4 *Let $\mathcal{A}_0 = (V, \mu_0)$ be a commutative associative algebra and $\mathcal{A}_t = (V, \mu_t)$ be a deformation of $\mathcal{A}_0$. Then*

$$\circlearrowleft_{x,y,z} \delta^2\mu_2(x, y, z) - \circlearrowleft_{x,z,y} \delta^2\mu_2(x, z, y) = 0.$$

Proof 13 *Using lemma 13.2 and the commutativity of μ_0, one has*

$$\begin{aligned}\circlearrowleft_{x,y,z} \delta^2\mu_2(x,y,z) - \circlearrowleft_{x,z,y} \delta^2\mu_2(x,z,y) &= \circlearrowleft_{x,y,z} \mu_2\circ\mu_0(x,y,z) - \circlearrowleft_{x,z,y} \mu_2\circ\mu_0(x,z,y)\\ &= \mu_2(x,\mu_0(y,z)) - \mu_2(\mu_0(x,y),z)\\ &+ \mu_0(y,\mu_2(z,x)) - \mu_0(\mu_2(y,z),x)\\ &+ \mu_2(z,\mu_0(x,y)) - \mu_2(\mu_0(z,x),y)\\ &- \mu_2(x,\mu_0(z,y)) + \mu_2(\mu_0(x,z),y)\\ &- \mu_2(z,\mu_0(y,x)) + \mu_2(\mu_0(z,y),x)\\ &- \mu_2(y,\mu_0(x,z)) - \mu_2(\mu_0(y,x),z)\\ &= 0.\end{aligned}$$

Theorem 13.5 *Let $\mathcal{A}_0 = (V,\mu_0)$ be a commutative associative algebra and $\mathcal{A}_t = (V,\mu_t)$ be a deformation of $\mathcal{A}_0$. Consider the bracket defined for $x,y \in V$ by $\{x,y\} = \mu_1(x,y) - \mu_1(y,x)$ where μ_1 is the first order element of the deformation μ_t.*

Then $(V,\mu_0,\{-,-\})$ is a Poisson algebra.

Proof 14 *The bracket is skewsymmetric by definition. The Leibniz relation follows from Lemma 13.3. Let us prove that the Jacobi condition is satisfied by the bracket. One has*

$$\begin{aligned}\circlearrowleft_{x,y,z} \{x,\{y,z\}\} &= \circlearrowleft_{x,y,z} (\mu_1(x,\mu_1(y,z)) - \mu_1(x,\mu_1(z,y)) +\\ &\quad -\mu_1(\mu_1(y,z),x) + \mu_1(\mu_1(z,y),x))\\ &= \mu_1(x,\mu_1(y,z)) - \mu_1(x,\mu_1(z,y)) +\\ &\quad -\mu_1(\mu_1(y,z),x) + \mu_1(\mu_1(z,y),x)\\ &\quad +\mu_1(y,\mu_1(z,x)) - \mu_1(y,\mu_1(x,z)) +\\ &\quad -\mu_1(\mu_1(z,x),y) + \mu_1(\mu_1(x,z),y)\\ &\quad +\mu_1(z,\mu_1(x,y)) - \mu_1(z,\mu_1(y,x)) +\\ &\quad -\mu_1(\mu_1(x,y),z) + \mu_1(\mu_1(y,x),z)\\ &= \circlearrowleft_{x,y,z} \mu_1\circ\mu_1(x,y,z) - \circlearrowleft_{x,z,y} \mu_1\circ\mu_1(x,z,y).\end{aligned}$$

The deformation equation (13.2.2) *implies for $s = 2$ that*

$$\mu_1\circ\mu_1 = -\mu_2\circ\mu_0 - \mu_0\circ\mu_2$$

which is equivalent to

$$\mu_1\circ\mu_1 = -\delta^2\mu_2$$

Then using the previous Lemmas

$$\circlearrowleft_{x,y,z} \{x,\{y,z\}\} = \circlearrowleft_{x,y,z} -\delta^2\mu_2(x,y,z) + \circlearrowleft_{x,z,y} \delta^2\mu_2(x,z,y) = 0$$

Remark 13.3 *The deformation quantization problem is the inverse problem : Given a Poisson algebra $\mathcal{A}$, find a formal deformation returning the original Poisson algebra structure on $\mathcal{A}$ in the quasi-classical limit. The deformed multiplication is called star-product.*

13.6 Deformations of unital algebras

In this section, we study the deformations of unital algebras. The main result is that a deformation of unital algebra is always unital.

Let $\mathcal{A} = (V, \mu_0)$ be a unital associative algebra with unit $1 \in V$, that is $\mu_0(x, 1) = \mu_0(1, x) = x$ for all $x \in V$. In Hopf language, it means that there exit a linear map $\eta : \mathbb{K} \to V$ such that $\mu \circ (\eta \otimes id) = \mu \circ (id \otimes \eta) = id$.

We have the following fundamental result:

Theorem 13.6 *1. The unit 1 of $\mathcal{A}$ is also the unit element of the formal deformation $\mathcal{A}_t = (V, \mu_t)$ of $\mathcal{A}$ if and only if*

$$\mu_n(x, 1) = \mu(1, x) = 0 \quad \forall n > 0, x \in \mathcal{A}. \tag{13.6.1}$$

In Hopf algebra language, if η is the unit for $\mathcal{A}$, then the unit η_t of the deformation $\mathcal{A}_t$ is defined by

$$\eta_t(\sum_n a_n t^n) = \sum_n a_n 1 t^n.$$

2. *If $\mathcal{A}$ is a unital algebra then every invertible element in $\mathcal{A}$ is invertible in $\mathcal{A}_t$.*

3. *Every formal deformation $\mathcal{A}_t = (V, \mu_t)$ of a unital algebra $\mathcal{A}$ is equivalent to formal deformation $\mathcal{A}_t = (V, \mu'_t)$ with the same unit as $\mathcal{A}$.*

Proof 15 *The first assertion follows from a direct calculation. The element 1 is a unit for μ_t if*

$$\mu_t(x, 1) = x = x + \sum_{n>0} \mu_n(x, 1) t^n.$$

By identification, we obtain $\mu_n(x, 1) = 0 \quad \forall n > 0, x \in \mathcal{A}$.

We prove now the second assertion:

Let a be an invertible element in $\mathcal{A}$. We consider the endomorphisms v_t and w_t, over $\mathbb{K}[[t]]$, defined for $x \in \mathcal{A}$ by

$$v_t(x) = \mu_t(x, a) \quad w_t(x) = \mu_t(a, x).$$

The endomorphisms v_t and w_t are invertible since v_0 and w_0 are invertible.

Assertion (3):

Two deformations are equivalent if there is a formal isomorphism f_t which is a $\mathbb{K}[[t]]$-linear map that may be written in the form

$$f_t = Id + tf_1 + t^2 f_2 + \cdots \qquad where \quad f_i \in End_{\mathbb{K}}(V)$$

such that (13.4.1) holds, which is equivalent to

$$\mu'_n(x,y) = \mu_n(x,y) + f_n(\mu_0(x,y)) - \mu_0(f_n(x),y) - \mu_0(x,f_n(y)) \qquad (13.6.2)$$

On another hand, we consider the n^{th} deformation equation of (13.2.1),

$$\sum_{i+j=n} \mu_i(\mu_j(x,y),z) - \mu_i(x,\mu_j(y,z)) = 0$$

in which we set $y = z = 1$ respectively $x = y = 1$ and $z = x$. Then we have

$$\begin{aligned} \mu_n(x,1) &= \mu_0(x,\mu_n(1,1)) \\ \mu_n(1,x) &= \mu_0(\mu_n(1,1),x) \end{aligned} \qquad (13.6.3)$$

We consider the formal isomorphism satisfying

$$f_1(1) = \mu_1(1,1), \quad f_n = 0 \text{ for } n \geq 2.$$

Using (13.6.2) and (13.6.3), the equivalent multiplication leads to a new deformed multiplication satisfying

$$\begin{aligned} \mu'_1(x,1) &= \mu_1(x,1) + f_1(\mu_0(x,1)) - \mu_0(f_1(x),1) - \mu_0(x,\mu_1(1,1)) \\ &= \mu_1(x,1) + f_1(x) - f_1(x) - \mu_1(x,1) = 0 \end{aligned}$$

Similarly, we obtain $\mu'_1(1,x) = 0$. By induction on n, we show that for all n, $\mu'_n(1,x) = \mu'_n(x,1) = 0$. Indeed, we assume $\mu'_k(1,x) = \mu'_k(x,1) = 0$ for $k = 1,\cdots,n-1$. We consider the isomorphism f_t satisfying $f_n(1) = \mu_n(1,1)$ and $f_k = 0\ \forall k \neq n$. Then, using (13.6.2) and (13.6.3), we obtain $\mu'_n(1,x) = \mu'_n(x,1) = 0$.

Observe that the product $(1 + f_n t^n)\cdots(1 + f_n t^n)$ converge when n tends to infinity

13.7 Deformations of Hopf Algebras

Given a Hopf algebra $H = (V,\mu_0,\Delta_0,\eta_0,\varepsilon_0,S_0)$ on the $\mathbb{K}$-vector space V, a deformation of H is a one parameter family $H_t = (V[[t]],\mu_t,\Delta_t,\eta_t,\varepsilon_t,S_t)$ where $V[[t]]$ a $\mathbb{K}[[t]]$-module.

By $k[[t]]$-linearity the morphisms μ_t, Δ_t, S_t are determined by their restrictions to $V \otimes V$ then

$$\mu_t : \begin{array}{l} V \otimes V \to V[[t]] \\ (x, y) \to \mu_t(x, y) = \sum_{m=0}^{\infty} \mu_m(x, y) t^m \end{array} \text{ with } \mu_m \in Hom(V \otimes V, V)$$

$$\Delta_t : \begin{array}{l} V \to V[[t]] \otimes V[[t]] = (V \otimes V)[[t]] \\ x \to \Delta_t(x) = \sum_{m=0}^{\infty} \Delta_m(x) t^m \end{array} \text{ with } \Delta_m \in Hom(V, V \otimes V)$$

$$S_t : \begin{array}{l} V \to V[[t]] \\ x \to S_t(x) = \sum_{m=0}^{\infty} S_m(x) t^m \end{array} \text{ with } S_m \in Hom(V, V)$$

Assuming $\eta_t(t) = t$ and $\varepsilon_t(t) = t$,we can write

$$\eta_t : \begin{array}{l} V \to \mathbb{K}[[t]] \\ x \to \eta_t(x) = \sum_{m=0}^{\infty} \eta_m(x) t^m \end{array} \qquad \varepsilon_t : \begin{array}{l} \mathbb{K} \to V \\ x \to \eta_t(x) = \sum_{m=0}^{\infty} \varepsilon_m(x) t^m \end{array}$$

We have the two important results

Proposition 13.4 *The unit and counit are preserved by deformation.*

Proof 16 *Follows from Theorem 13.6.*

Theorem 13.7 *A deformation of Hopf algebra as a bialgebra is automatically a Hopf algebra.*

Proof 17 *Recall that $End_{\mathbb{K}}V$ is naturally a unital $\mathbb{K}$-algebra in which the multiplication is given by $f \star g = \mu(f \otimes g)\Delta$ and the unit is $\eta\varepsilon$, the composite of the counit and the unit of H. An antipode is then the two sided inverse for the identity map in $End_{\mathbb{K}}V$. Now, if H_t is a deformation of H, then the $\mathbb{K}[[t]]$-submodule $(End_{\mathbb{K}}V[[t]])$ of $End_{\mathbb{K}[[t]]}V[[t]] = End_{\mathbb{K}[[t]]}V$ is , in fact, a subalgebra (with the same unit) and is therefore a deformation of $End_{\mathbb{K}}V$. As the invertibility of an element in an algebra is preserved in every deformation (see 13.6), H_t has an antipode whenever H has one (the antipode is not necessarily the same).*

It follows that a deformation of $H = (V, \mu_0, \Delta_0, \eta_0, \varepsilon_0, S_0)$ can be written $H_t = (V[[t]], \mu_t, \Delta_t, \eta_0, \varepsilon_0, S_0)$. Then we can consider the deformation as a pair of deformations (μ_t, Δ_t) which together give on $V[[t]]$ the structure of bialgebra over $\mathbb{K}[[t]]$. ie

- μ_t is associative
- Δ_t is coassociative

- $(\mu_t \otimes \mu_t)\tau\left(\Delta_t \otimes \Delta_t\right) = \Delta_t\mu_t$ where $\tau\left(x_1 \otimes x_2 \otimes x_3 \otimes x_4\right) = x_1 \otimes x_3 \otimes x_2 \otimes x_4$.

Remark 13.4 *1. As in the algebra case, every infinitesimal deformation* (μ_1, Δ_1) *defines a cocycle, and conversely.*

2. If two cocycles are cohomologous, then the corresponding infinitesimal deformations are equivalent by an infinitesimal automorphism.

3. The obstruction to extending a deformation of order m *to a deformation of order* $m+1$ *is an element of* $H^2\left(\widehat{C}\right)$

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