Algèbres de Lie symplectiques

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All vector space considered are finite dimensional over a commutative field $\mathbb{K}$ of characteristic zero if not said otherwise.

Définition

Let $\mathfrak{g}$ be a vecor space and $[\,.,.\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ a bilinear map. $(\mathfrak{g}, [\,.,\,.])$ is called a Lie algebra if:

1. $[x, y] = -[y, x]$ (which equivalent to $[x, x] = 0$), $\forall x, y \in \mathfrak{g}$ (skew-symmetry);
2. $[x, [y, z]] + [x, [y, z]] + [x, [y, z]] = 0$, $\forall x, y, z \in \mathfrak{g}$ (Jacobi identity).

Examples. If Let $(\mathcal{A}, .)$ be an associative algebra. The underlying vector space $\mathcal{A}$ with the new products:

$$[x, y] := x.y - y.x, \ \ \forall\, x, y \in \mathcal{A},$$

is a Lie algebra. We denote $\mathcal{A}^-$ this Lie algebra.
If V is a vector space, we consider $(\text{End}(V), \circ)$ the associative algebra of the endomorphism of V (where $\circ$ is the composition of maps). The Lie algebra $((\text{End}(V))^-$ is denoted by $\mathfrak{g}l(V)$.
If $n \in \mathbb{N}^\star$, we denote by $\mathfrak{g}l(n)$ the Lie algebra $(M_n(\mathbb{K}))^-$ of tha associtive algebra $((M_n(\mathbb{K}), .)$ of matrices $n \times n$ (with coefficients in $\mathbb{K}$), where $''.''$ is the multiplication of matrices.

Notation.
Let $(\mathfrak{g}, [\,.,\,.])$ be a Lie algebra and V, W two sub-vector spaces of $\mathfrak{g}$.
We denote by $[V, W]$ the sub-vector space of $\mathfrak{g}$ spanned by $\{[v, w],\ v \in V, w \in W\}$.

Définition

Let $(\mathfrak{g}, [\,.,\,.])$ be a Lie algebra and S a non-empty subset of S.

1. S is called an ideal of $\mathfrak{g}$ if $[\mathfrak{g}, S] \subseteq S$.
2. S is called a Lie subalgebra of $\mathfrak{g}$ if $[S, S] \subseteq S$.
3. The derived series of $\mathfrak{g}$ is defined by:
$$\mathfrak{g}^{(0)} := \mathfrak{g},\ \ \mathfrak{g}^{(n+1)} := [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}],\ \ \forall n \in \mathbb{N}^\star.$$
For all $n \in \mathbb{N}$, $\mathfrak{g}^{(n)}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g}^{(n+1)} \subseteq \mathfrak{g}^{(n)}$.
4. The lower central series or descending central series of $\mathfrak{g}$ is defined by:
$$\mathfrak{g}^{1} := \mathfrak{g},\ \ \mathfrak{g}^{n+1} := [\mathfrak{g}^{n}, \mathfrak{g}^{n}],\ \ \forall n \in \mathbb{N}^\star.$$
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Définition A Lie algebra $\mathfrak{g}$ is called solvable (resp. nilpotent) if there exists $n \in \mathbb{N}$ (resp. $n \in \mathbb{N}^\star$) such that $\mathfrak{g}^{(n)} = \{0\}$ (resp. $\mathfrak{g}^n = \{0\}$).

Définition

Let $(\mathfrak{g}, [\,.,.\,])$ and $(\mathfrak{g}', [\,.,.\,]')$be two Lie algebra and $f : \mathfrak{g} \to \mathfrak{g}'$ a linear map.
f is called a morphism of Lie algebras if $f([x,y]) = [f(x), f(y)]', \ \forall\, x, y \in \mathfrak{g}$. If, in addition, f is invertible, f is called an isomorphism of Lie algebras.

Définition

Let $(\mathfrak{g}, [\,.,.\,])$ be a Lie algebra and f an endomorphism of underlying vector space of $\mathfrak{g}$.

1. $\mathfrak{g}^{(1)} := [\mathfrak{g}, \mathfrak{g}]$ is called the derived ideal of $\mathfrak{g}$.
2. $\mathfrak{z}(\mathfrak{g}) := \{x \in \mathfrak{g} \ / \ [x, y] = 0, \ \forall\, y \in \mathfrak{g}\}$ is an ideal of $\mathfrak{g}$ called the center of $\mathfrak{g}$.
3. f is called a morphism of Lie algebras (or an endomorphism of the Lie algebra $\mathfrak{g}$ if

$$f([x,y]) = [f(x), f(y)], \ \forall\, x, y \in \mathfrak{g}.$$

4. f is called a derivation of $\mathfrak{g}$ if

$$f([x,y]) = [f(x), y] + [x, f(y)], \ \forall x, y \in \mathfrak{g}.$$

It is easy to verify that if f and h are two derivations of $\mathfrak{g}$, then $[f, h] := f \circ h - h \circ f$ is still a derivation of $\mathfrak{g}$.
If we denote by Der$(\mathfrak{g})$ the set of all derivations of $\mathfrak{g}$, then $(\text{Der}(\mathfrak{g}), [\ ,\ ])$ is a Lie algebra. called the derivation algebra of the Lie algebra $\mathfrak{g}$.

Définition

Let $(\mathfrak{g}, [.,.])$ be a Lie algebra, V is vector space and $\pi : \mathfrak{g} \to \mathfrak{g}l(V)$.
π is called a representation of $\mathfrak{g}$ on V if π is a morphism of Lie algebras (or V is a $\mathfrak{g}$-module), i.e. $\pi([x, y]) = \pi(x)\pi(y) - \pi(y)\pi(x), \ \forall\, x, y \in \mathfrak{g}$.
We denote $\pi(x)(v) =: x.v,$ for all $x \in \mathfrak{g}$ and $v \in V$.
π is called trivial (or the $\mathfrak{g}-$module V is trivial) if $\pi(x) = 0, \ \forall x \in \mathfrak{g}$.

Définition

Let $(\mathfrak{g}, [.,.])$ be a Lie algebra. Let $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ a skew-symmetric bilinear form of $\mathfrak{g}$.

1. ω is called a scalar $2-$cocycle of $\mathfrak{g}$ if
$$\omega(x, [y, z]) + \omega(y, [z, x]) + \omega(z, [x, y]) = 0, \quad \forall\, x, y, z \in \mathfrak{g}.$$
2. ω is called a scalar 2-coboundary of $\mathfrak{g}$ if there exists f a linera form of $\mathfrak{g}$ such that
$$\omega(x, y) = f([x, y]), \quad \forall x, y \in \mathfrak{g}.$$

Denote as usual by $Z^2(\mathfrak{g}, \mathbb{K})$ the vector space of the scalar $2-$cocycles of $\mathfrak{g}$ and by $B^2(\mathfrak{g}, \mathbb{K})$ the vector space of the scalar $2-$coboundaries of $\mathfrak{g}$. We define the scalar second cohomology space of $\mathfrak{g}$ as the quotient vector space $H^2(\mathfrak{g}, \mathbb{K}) := Z^2(\mathfrak{g}, \mathbb{K})/B^2(\mathfrak{g}, \mathbb{K})$.

Same definitions if we repalce $\mathbb{K}$ a trivial $\mathfrak{g}-$module (we change scalar by with values in V).

Définition
Let $(\mathfrak{g}, [\,.,.\,])$ be a Lie algebra. If there exists a non-degenerate $\omega \in Z^2(\mathfrak{g}, \mathbb{K})$, we will said that $(\mathfrak{g}, \omega)$ is a symplectic Lie algebra and ω is a symplectic structure on Lie algebra $\mathfrak{g}$.In this case the dimension of $\mathfrak{g}$ is even.

Example. Consider the $2-$dimensional Lie algebra $\mathfrak{g} :=$ span$\{a, b\}$ where the product is defined by $[a, b] = b$. The skew-bilinear form ω on $\mathfrak{g}$ defined by $\omega(a, b) = 1$ is a symplectic structure on $\mathfrak{g}$.

Some extensions.
1. Direct product of Lie algebras.
Let $(\mathfrak{g}, [\,.,.\,])$ and $(\mathfrak{g}', [\,.,.\,]')$ be two Lie algebras. The direct product of this two Lie algebras is the vector space $\mathfrak{g} \times \mathfrak{g}'$ with the Lie algebra structure defined by the product

$$\{x + x', y + y'\} := [x, y] + [x', y']', \ \ \forall x, y \in \mathfrak{g}, \forall x', y' \in \mathfrak{g}'.$$

2. Central extensions of Lie algebras.
Let $(\mathfrak{g}, [\,.,.\,]_{\mathfrak{g}})$ be a Lie algebra, V a vector space and $\omega : \mathfrak{g} \times \mathfrak{g} \to V$ a bilinear map. On the vector space $\mathfrak{g} \times V$, we consider the following product

$$[(x, v), (y, w)] := [x, y]_{\mathfrak{g}} + \omega(x, y), \quad \forall (x, v), (y, w) \in \mathfrak{g} \times V.$$

$(\mathfrak{g} \times V, [\,.,.\,])$ is a Lie algebra if and only if ω is a $2-$cocycle of $\mathfrak{g}$ with values in V (V is considered as a trivial $\mathfrak{g}-$module).

3. Semi-direct product of Lie algebras.
Let $(\mathfrak{g}_1, [\,.,.\,]_1)$ and $(\mathfrak{g}_2, [\,.,.\,]_2)$ two Lie algebras. Let $\pi : \mathfrak{g}_1 \to \mathrm{Der}(\mathfrak{g}_2)$ a representation of Lie algebras.
The vector space $\mathfrak{g}_1 \times \mathfrak{g}_2$ with the following product is a Lie algebra

$$[(x_1, x_2), (y_1, y_2)] := ([x_1, y_1]_1, [x_2, y_2]_2 + \pi(x_1)(y_2) - \pi(y_1)(x_2)),$$

$\forall\, (x_1, x_2), (y_1, y_2) \in \mathfrak{g}_1 \times \mathfrak{g}_2$.
This Lie algebra is called the semi-direct product of $\mathfrak{g}_2$ by $\mathfrak{g}_1$ by means of π and it is denoted $\mathfrak{g}_1 \ltimes_\pi \mathfrak{g}_2$.

Examples.
1. Heisenberg Lie algebra of dimension 3: $H_3 := \text{span}\{x, y, z\}$ with th eparoduct defined by:

$$[x, y] = z$$

This Lie algebra is the central extension of the abelian Lie algebra $\mathfrak{g} :=$ x,y by means of the $2-$cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}z$ defined by $\omega(x, y) = z$.
2. Now we consider de derivation D of H_3 defined by

$$D(x) = y, \ D(y) = -x, \ D(z) = 0.$$

Let us consider the one-dimensional Lie algebra $\mathbb{K}d$ and the representation of Lie algebra $\pi : \mathbb{K} \to \text{Der}(H_3)$ defined by $\pi(d) := D$. The semi-product of $\mathbb{K}d$ by H_3 by means of π is called the oscillator Lie algebra of dimension 4. The product on this algebra $O_4 := H_3 \ltimes_\pi \mathbb{K}d$ is defined by:

$$[d, x] = y, \ [d, y] = -x, \ [x, y] = z.$$

Construction of symplectic Lie algebras.

Let us consider $(A, .)$ be a commutative associative algebra (example: $A := \mathbb{R}^n$, where $n \in \mathbb{N}^\star$). The vector space $A \oplus A$ with the product

$$[(a, b), (a', b')] := (0, a.b' - a'.b), \ \forall a, b, a', b' \in A,$$

is the Lie algebra called affine algebra associted to commutative associative algebra denoted aff(A).
If A admits the symmetric nondegenerate associative bilinear form B (Recall that B is associative if $B(a.b, c) = B(a, b.c), \ \forall a, b, c \in A)$, then the following bilinear form ω is a symplectic structure on aff(A):

$$\omega((a, b), (a', b')) := B(a, b') - B(a', b), \ \forall a, b, a', b' \in A,$$

.

Now let us consider a commutative associative algebra $(S, .)$. The vector space $A := S \oplus S^\star$ with the product defined by

$$(a+f) \bullet (b+h) := a.b + f \circ M_a + h \circ M_b, \forall (a, f), (b, h) \in S \times S^\star,$$

where $M_x : S \to S$ defined by $M_x(y) := x.y, \forall x, y \in S,$
is a commutative associative algebra. In addition, the bilinear form B of $S \oplus S^\star$ defined by

$$B(a+f, b+h) := f(b) + h(a), \forall (a, f), (b, h) \in S \times S^\star,$$

is symmetric, nondegenerate and associative.
Consequently, aff$(S \oplus S^\star)$ is a symplectic Lie algebra. The Lie structure is defined by

$$[(a+f, b+h), (a'+f', b'+h')] := (0, (a.b' - a'.b) + (f \circ M_{b'} + h' \circ M_a - f' \circ M_b - h \circ M_{a'}),$$

$\forall (a+f), (b+h), (a'+f'), (b'+h') \in S \oplus S^\star.$
The symplectic structure on aff$(S \oplus S^\star)$ is defined by

$$\omega((a+f, b+h), (a'+f', b'+h')) := (h'(a) - f'(b)) + (f(b') - h(a')),$$

$\forall (a+f), (b+h), (a'+f'), (b'+h') \in S \oplus S^\star.$

Left-symmetric algebra structure on the underlying vector space of a symplectic Lie algebra.

Let $(\mathfrak{g}, [\,.,.\,]\omega)$ a symplectic Lie algebras.
Let $x, y \in \mathfrak{g}$. We consider the linear form $\varphi_{x,y} : \mathfrak{g} \to \mathbb{K}$ defined by:
$\varphi_{x,y}(z) := \omega([x,z], y),\ \forall z \in \mathfrak{g}$. The fact that ω is non-degenerate implies that there exists an unique element of $\mathfrak{g}$,denoted $a(x,y)$, such that
$\varphi_{x,y}(z) = \omega(a(x,y), z), \forall z \in \mathfrak{g}$. Consequently, $a : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear map such that

$$\omega(a(x,y), z) = -\omega(y, [x,z]),\ \ \forall\, x, y, z \in \mathfrak{g}.$$

Let $x, y, z, t \in \mathfrak{g}$,
$\omega(x,[y,z]) + \omega(y,[z,x]) + \omega(z,[x,y]) = 0 \iff \omega([x,y] - a(x,y) - a(y,x), z) = 0$
and
$0 = \omega(t, [x,[y,t]] + [y,[z,x]] + [z,[x,y]]) =$
$\omega(a(a(x,y),t) - a(a(y,x),t) - a(x, a(y,t) + a(y, a(x,t), z)$.
Therefore, forall $x, y \in \mathfrak{g}$:

1. $[x,y] = a(x,y) - a(y,x)$;
2. $a(a(x,y),t) - a(x, a(y,t) = a(a(y,x),t) - a(y, a(x,t))$.

We conclude that the product "a" defines on the underlying vector space on $\mathfrak{g}$ a structure of Left-symmetric algeabra where the associated Lie algebra is the Lie algebra of the start.

Définition Let $(A, .)$ be a non-associative algebra. The associator is the trilinear map Asso : $\mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ defined by

$$\text{Asso}(x, y, z) = (x.y).z - x.(y.z), \quad \forall\, x, y, z \in \mathcal{A}.$$

1. The algebra $(\mathcal{A}, .)$ is called an associative algebra if the associator is identically zero.
2. The algebra $(\mathcal{A}, .)$ is called a left-symmetric algebra (resp. right-symmetric algebra) if
$$\text{Asso}(x, y, z) = \text{Asso}(y, x, z), \quad \forall\, x, y, z \in \mathcal{A}.$$
$$(\text{resp. } \text{Asso}(x, y, z) = \text{Asso}(x, z, y), \quad \forall\, x, y, z \in \mathcal{A}).$$

Définition Let $(A, .)$ be a left-symmetric algebra. Then the underlying vector space of A with the new product:

$$[x, y] := x.y - y.x, \ \forall x, y \in A,$$

is a Lie algebra, denoted A^- (i.e. a left-symmetric algebra is Lie-admissible).

Proposition.

Let $(A,.)$ be a left symmetric algebra $\iff$ $L : A^- \to \mathfrak{gl}(A)$ defined by: $L(a) := L_a, \forall\, a \in A,$ is a representation of the Lie algebra A^- . This representation define a structure of A^-module on the underlying vector space of A.

Where $L(a)(b) := a.b, \quad \forall a, b \in A$. ($L_a$ is called the left-multiplication by a in A.)

Recall that if $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of a Lie algebra $\mathfrak{g}$ in the vector space V. The dual representation $\pi^\star : \mathfrak{g} \to \mathfrak{gl}(V^\star)$ of π is defined by:

$$(\pi^\star(x)(f))(y) := -f(\pi(x)(y)), \quad \forall\, x, y \in \mathfrak{g}.$$

Construction symplectic Lie algebras from left-symmetric algebras.

Let $(A,.)$ be a left symmetric algebra.
Consider $\mathfrak{g} := A^- \ltimes_{L^\star} A^\star$ the semi-direct product of the abelian Lie algebra $A^\star$ by A^- by means of $L^\star$.
The product of this Lie algebra is defined by:

$$[x + f, y + h] = (x.y - y.x) - h \circ L_x + f \circ L_y, \quad \forall\, (x + f, y + h) \in A \times A^\star.$$

A symplectic Lie algebra structure ω on $\mathfrak{g}$ is defined by

$$\omega(x + f, y + h) = h(x) - f(y), \quad \forall\, (x + f, y + h) \in A \times A^\star.$$

How concretely to construct examples of symplectic Lie algebras from this construction?

We start with this proposition which gives us left symmetric structures on Lie algebras which have invertible derivations. Note that according to a theorem of N. Jacobson from the 1950s, any Lie algebra which has at least one invertible derivation is necessarily a nilpotent Lie algebra.

Proposition.

Let $(\mathfrak{g}, [\,,\,])$ be a Lie algebra with an invertible deivation D.
Then the vector space $\mathfrak{g}$ with the following multiplication "$\star$" is a left symmetric algebra:

$$x \star y := [D^{-1}(x), y], \quad \forall\ x, y \in \mathfrak{g}.$$

Let us consider a vector space A of dimension n with a basis $\{e_1, \ldots, e_n\}$. On this vector space, we consider the bilinear form $\circ : A \times A \to A$ defined by

$$e_i \circ e_j = e_{i+j} \text{ if } i + j \leq n \text{ or } 0 \text{ if } i + j > n.$$

$(A, \circ)$ is a nilpotent associative commutative algebra isomorphic to $X\mathbb{K}[X]/X^{n+1}\mathbb{K}[X]$. It is clear that the endomorphism δ of A defined by: $\delta(e_i) := ie_i, \ \forall\, i \in \{1, \ldots, n\}$ is an invertible derivation of $(A, \circ)$. Now if $(\mathfrak{g}, [\ ,\ ]_\mathfrak{g})$ is a Lie algebra, then the vector space $\mathfrak{g} \otimes A$ with the product (bilinear map) defind by :

$$[x \otimes a, y \otimes b] = [x, y]_\mathfrak{g} \otimes (a \circ b), \ \forall\, x, y \in \mathfrak{g}, a, b \in A,$$

is a nilpotent Lie algebra and the endomorphism D of the vector space $\mathfrak{g} \otimes A$ defined by

$$D(x \otimes a) := x \otimes \delta(a) \ \forall\, x \in \mathfrak{g}, a \in A,$$

Is an invertible derivation of the Lie algebra $(g \otimes A, [\ ,\ ])$. Consequently, the vector space $g \otimes A$ with the following product (bilinear map) $\star$ is a Left symmetric algebra:

$$(x \otimes a) \star (y \otimes b) = [D^{-1}(x \otimes a), y \otimes b], \ \forall\, x, y \in \mathfrak{g}, a, b \in A.$$

If we denote by $\mathfrak{g}(A)$ this Left symmetric algebra $(\mathfrak{g} \otimes A, \star)$, then with a simple caculus, we can write the Lie structure and the symplectic structure of the Lie algebra $\mathfrak{g}(A)^- \ltimes (\mathfrak{g}(A))^\star$.

Double extensions of symplectic Lie algebras

There is a procedure for constructing symplectic Lie algebras called the symplectic double extension procedure. This procedure was introduced by Alberto Medina and Philippe Revoy. In the following we will recall this process

Proposition. Let $(\mathfrak{g}, \omega)$ be a symplectic Lie algebra and D a derivation of $\mathfrak{g}$. Then the bilinear form $\varphi_D : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ defined by

$$\varphi_D(x, y) := \omega(D(x), y) + \omega(x, D(y)) = \omega((D + D^\star)(x), y), \quad \forall x, y \in \mathfrak{g},$$

is an element of $Z^2(\mathfrak{g}, \mathbb{K})$ (i.e φ is a scalar $2-$cocycle of $\mathfrak{g}$), where $D^\star$ is the transpose of D with respect to ω.

Proof. φ_D is a skew-symmetric because ω is skew-symmetric.

Let $x, y, z \in \mathfrak{g}$,

$\varphi_D(x, [y, z]) + \varphi_D(y, [z, x]) + \varphi_D(z, [x, y])$
$= \omega(D(x), [y, z]) + \omega(x, D([y, z]) + \omega(D(y), [z, x]) + \omega(y, D([z, x]) + \omega(D(z), [y, z]) + \omega(z, D([x, y])$
$= \omega(D(x), [y, z]) + \omega(x, [D(y), z]) + \omega(x, [y, D(z)]) + \omega(D(y), [z, x]) + \omega(y, [D(z), x]) + \omega(y, [z, D(x)]) + \omega(D(z), [x, y]) + \omega(z, [D(x), y]) + \omega(z, [x, D(y)])$
$= \omega(D(x), [y, z]) + \omega(z, [D(x), y]) + \omega(y, [z, D(x)]) + \omega(D(y), [z, x]) + \omega(x, [D(y), z]) + \omega(z, [x, D(y)]) + \omega(D(z), [x, y]) + \omega(y, [D(z), x]) + \omega(x, [y, D(z)])$
$= 0 + 0 + 0 = 0$ (because ω is a $2'$ cocycle).

We conclude that φ_D is a scalar $2-$cocycle.

Proposition. Let $(\mathfrak{g},\omega)$ be a symplectic Lie algebra and D a derivation of $\mathfrak{g}$. Then the bilinear form $\Omega_D : \mathfrak{g}\times\mathfrak{g}\to\mathbb{K}$ defined by

$$\Omega_D(x,y) := \omega(((D+D^\star)D + D^\star(D+D^\star))(x),y), \quad \forall x,y\in\mathfrak{g},$$

is an element of $Z^2(\mathfrak{g},\mathbb{K})$ (i.e Ω_D is a scalar $2-$cocycle of $\mathfrak{g}$), where $D\star$ is the transpose of D with respect to ω.

Proof. Same calculations as in the last proof.

Let $(\mathfrak{g}, [\,.,.\,]_{\mathfrak{g}}, \omega)$ be a symplectic Lie algebra and D a derivation of $\mathfrak{g}$.
We consider the two scalar cocycles of $\mathfrak{g}$.

1. $2-$cocycles $\varphi_D : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ defined by:
   $\varphi_D(x,y) := \omega(D(x),y) + \omega(x,D(y)) = \omega((D+D^{\star})(x),y), \quad \forall x,y \in \mathfrak{g},$
2. $\Omega_D : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ defined by:
   $\Omega_D(x,y) := \omega(((D+D^{\star})D + D^{\star}(D+D^{\star}))(x),y), \quad \forall x,y \in \mathfrak{g}.$

First extension: central extension of $\mathfrak{g}$.

Let us consider the one-dimensional Lie algebra $\mathbb{K}e$ and $\mathfrak{g}'$ the central extension of $\mathfrak{g}$ by $\mathbb{K}e$ by means of φ_D. Then

1. $\mathfrak{g}' = \mathfrak{g} \oplus \mathbb{K}e;$
2. the product of $\mathfrak{g}'$ is defined by:

$$[x+\alpha e, y+\beta e]_{\mathfrak{g}'} := [x,y]_{\mathfrak{g}} + \varphi_D(x,y)\, e,$$

$\forall\, x,y \in \mathfrak{g}, \forall \alpha,\beta \in \mathbb{K}.$

Hypothesis: $\Omega_D \in B^2(\mathfrak{g}, \mathbb{K})$.
Then there exists $f \in \mathfrak{g}^\star$ such that:

$$\forall x, y \in \mathfrak{g}, \quad \Omega_D(x, y) = f([x, y]_\mathfrak{g}).$$

Since ω is nondegenerate, **there exists an unique** $z_D \in \mathfrak{g}$ **such that**

$$f(x) := \omega_\mathfrak{g}(z_D, x), \quad \forall x \in \mathfrak{g}.$$

Now, let us consider $\delta$the endomorphism of the vector space underlying of g' defined by

$$\delta(e) := 0 \quad \text{and} \quad \delta(x) := -D(x) + \omega(z_D, x)e, \ \forall\, x \in \mathfrak{g}.$$

Claim: δ **is a derivation of the Lie algebra** $(\mathfrak{g}', [\,.,.\,]_{\mathfrak{g}'})$.

Second extension: Semi-direct product of the one-dimensional Lie algebra by the Lie algebra $\mathfrak{g}'$.

Let us consider the one-dimensional Lie algebra $\mathbb{K}d$ and the representaion $\pi : \mathbb{K}d \to \mathfrak{g}l(\mathfrak{g}')$ defined by: $\pi(d) := \delta$. We denote by $\tilde{g}$ the semi-direct product of $\mathbb{K}d$ by $\mathfrak{g}'$ by means of π (i.e. $\tilde{\mathfrak{g}} := \mathfrak{g}' \ltimes_\pi \mathbb{K}d$).

The underlying vector space of $\tilde{\mathfrak{g}}$ is:

$$\tilde{\mathfrak{g}} := \mathbb{K}e \oplus \mathfrak{g} \oplus \mathbb{K}d.$$

The product of $\tilde{\mathfrak{g}}$ is defined by

$$[x, y]_{\tilde{\mathfrak{g}}} := [x, y]_{\mathfrak{g}} + \omega((D + D^\star)(x), y)e, \ \ \forall\, x, y \in \mathfrak{g},$$

$$[d, x]_{\tilde{\mathfrak{g}}} := -D(x) - \omega(z_D, x)e, \quad \forall\; x \in \mathfrak{g}.$$

The Lie algebra $\tilde{\mathfrak{g}}$ admits the following symplectic structure $\tilde{\omega}$ defined by

$$\tilde{\omega}_{|\mathfrak{g}\times\mathfrak{g}} := \omega \quad \text{and} \quad \omega(e,d] := 1,$$

Other undefined products are zero.

The symplectic Lie algebra $(\tilde{\mathfrak{g}}, [\,.,.\,], \tilde{\omega})$ is called the symplectic double extension of $(\mathfrak{g}, \omega)$ by means (D, z_D).
Let us remark that e is in the center of $\tilde{\mathfrak{g}}$ and $\dim(\tilde{g}) = \dim(\mathfrak{g}) + 2$.

Theorem. Let $(\mathfrak{g}, \omega)$ be a symplectic Lie algebra which contains a one-dimensional isotropic ideal I. Then $(\mathfrak{g}, \omega)$ is a symplectic double extension of symplectic Lie algebra $(\mathfrak{g}', \omega')$.
Corollary. Let $(\mathfrak{g}, \omega)$ be a symplectic Lie algebra such that $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$. Then $(\mathfrak{g}, \omega)$ is a symplectic double extension of symplectic Lie algebra $(\mathfrak{g}', \omega')$.
Corollary.
Every nilpotent symplectic Lie algebra is obtained from $\{0\}$ by a finite sequence of symplectic double extensions.