# An introduction to symplectic mechanics: conservative and dissipative systems

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*Mechanics is the paradise of the mathematical sciences, because by means of it one comes to the fruits of mathematics*

Leonardo da Vinci

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# 1: Introduction to Hamiltonian and Lagrangian mechanics

## Different geometries provide different dynamics

## Dynamics on symplectic geometry

As it is well known, Hamiltonian dynamics are developed using symplectic geometry. Indeed, let $(M, \omega)$ be a symplectic manifold, that is, $\omega$ is a non–degenerate closed 2-form, say $d\omega = 0$ and $\omega^n \neq 0$, where $M$ has even dimension $2n$. Then, if $H : M \longrightarrow \mathbb{R}$ is a Hamiltonian function, the Hamiltonian vector field $X_H$ is obtained using the equation

$$\flat(X_H) = dH \tag{1}$$

where $\flat$ is the vector bundle isomorphism

$$\flat : TM \longrightarrow T^*M \, , \; \flat(v) = i_v \, \omega$$

In Darboux coordinates $(q^i, p_i)$ we have $\omega = dq^i \wedge dp_i$ and

$$X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i}$$

in such a way that an integral curve $(q^i(t), p_i(t))$ satisfies the Hamilton equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \, , \; \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \tag{2}$$

## Dynamics on cosymplectic geometry

A cosymplectic structure on an odd-dimensional manifold $M$ is a pair $(\Omega, \eta)$ where $\Omega$ is a closed 2-form, $\eta$ is a closed 1-form, and $\eta \wedge \Omega^n \neq 0$, where $M$ has dimension $2n+1$. $(M, \Omega, \eta)$ will be called a cosymplectic manifold.

There is a Darboux theorem for a cosymplectic manifold, that is, there are local coordinates (called Darboux coordinates) $(q^i, p_i, z)$ around any point of $M$ such that

$$\Omega = dq^i \wedge dp_i \ , \ \eta = dz$$

There exists a unique vector field (called Reeb vector field) $\mathcal{R}$ such that

$$i_{\mathcal{R}} \Omega = 0 \ , \ i_{\mathcal{R}} \eta = 1$$

In Darboux coordinates we have

$$\mathcal{R} = \frac{\partial}{\partial z}$$

Let $H : M \longrightarrow \mathbb{R}$ be a Hamiltonian function, say $H = H(q^i, p_i, z)$.

Consider the vector bundle isomorphism

$$\tilde{\flat} : TM \longrightarrow T^*M \; , \; \flat(v) = i_v \, \Omega + \eta(v) \, \eta$$

and define the gradient of $H$ by

$$\tilde{\flat}(\mathrm{grad}\; H) = dH$$

Then

$$\mathrm{grad}\; H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} + \frac{\partial H}{\partial z}\frac{\partial}{\partial z} \qquad (3)$$

Next we can define two more vector fields:

- The Hamiltonian vector field

$$X_H = \mathrm{grad}\; H - \mathcal{R}(H)\mathcal{R}$$

- and the evolution vector field

$$\mathcal{E}_H = X_H + \mathcal{R}$$

From (3) we obtain the local expression

$$\mathcal{E}_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} + \frac{\partial}{\partial z} \tag{4}$$

Therefore, an integral curve $(q^i(t), p_i(t), z(t))$ of $\mathcal{E}_H$ satisfies the time-dependent Hamilton equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \tag{5}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \tag{6}$$

$$\frac{dz}{dt} = 1 \tag{7}$$

and then $z = t + const$ so that both coordinates can be identified.

## Dynamics on contact geometry

Consider now a contact manifold $(M, \eta)$ with contact form $\eta$; this means that $\eta \wedge d\eta^n \neq 0$ and $M$ has odd dimension $2n+1$. There exists a unique vector field $\mathcal{R}$ (also called Reeb vector field) such that

$$i_{\mathcal{R}}\, d\eta = 0 \ , \ i_{\mathcal{R}}\, \eta = 1$$

There is a Darboux theorem for contact manifolds so that around each point in $M$ one can find local coordinates (called Darboux coordinates) $(q^i, p_i, z)$ such that

$$\eta = dz - p_i\, dq^i$$

In Darboux coordinates we have

$$\mathcal{R} = \frac{\partial}{\partial z}$$

Define now the vector bundle isomorphism

$$\bar{\flat} : TM \longrightarrow T^*M \ , \ \bar{\flat}(v) = i_v\, d\eta + \eta(v)\, \eta$$

For a Hamiltonian function $H$ on $M$ we define the Hamiltonian vector field by

$$\bar{\flat}(X_H) = dH - (\mathcal{R}(H) + H)\,\eta$$

In Darboux coordinates we get this local expression

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - (\frac{\partial H}{\partial q^i} + p_i\frac{\partial H}{\partial z})\,\frac{\partial}{\partial p_i} + (p_i\frac{\partial H}{\partial p_i} - H)\,\frac{\partial}{\partial z} \tag{8}$$

Therefore, an integral curve $(q^i(t), p_i(t), z(t))$ of $X_H$ satisfies the contact Hamilton equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \tag{9}$$

$$\frac{dp_i}{dt} = -(\frac{\partial H}{\partial q^i} + p_i\frac{\partial H}{\partial z}) \tag{10}$$

$$\frac{dz}{dt} = (p_i\frac{\partial H}{\partial p_i} - H) \tag{11}$$

## Example

Consider a Hamiltonian system given by the Hamiltonian

$$H(q,p,z) = \frac{p^2}{2m} + V(q) + \gamma\, z$$

where $\gamma$ is a constant.
We obtain the following dynamical equations

$$\dot{q} = \frac{p}{m}$$
$$\dot{p} = -\frac{\partial V}{\partial q} - \gamma\, z$$
$$\dot{z} = \frac{p^2}{2m} - V(q) - \gamma\, z$$

that are just the **damped Newtonian equations**.

## Symplectic Lagrangian formalism

Let $L : TQ \longrightarrow \mathbb{R}$ be a Lagrangian function, where $Q$ is a configuration $n$-dimensional manifold. Then, $L = L(q^i, \dot{q}^i)$, where $(q^i)$ are coordinates in $Q$ and $(q^i, \dot{q}^i)$ are the induced bundle coordinates in $TQ$. We will assume that $L$ is regular, that is, the Hessian matrix

$$\left( \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right)$$

is regular. Using the canonical endomorphism $S$ on $TQ$ locally defined by

$$S = dq^i \otimes \frac{\partial}{\partial \dot{q}^i}$$

one can construct a 1-form $\lambda_L$ defined by

$$\lambda_L = S^*(dL)$$

and the 2-form

$$\omega_L = -d\lambda_L$$

Then, $\omega_L$ is symplectic if and only if $L$ is regular.

Consider now the vector bundle isomorphism

$$\flat_L : T(TQ) \longrightarrow T^*(TQ)$$
$$\flat_L(v) = i_v \, \omega_L$$

and the Hamiltonian vector field

$$\xi_L = X_{E_L}$$

defined by

$$\flat_L(\xi_L) = dE_L$$

where $E_L = \Delta(L) - L$ is the energy, where $\Delta = \dot{q}^i \, \frac{\partial}{\partial \dot{q}^i}$ is the Liouville vector field on $TQ$. The vector field $\xi_L$ (the Euler-Lagrange vector field) is locally given by

$$\xi_L = \dot{q}^i \, \frac{\partial}{\partial q^i} + B^i \, \frac{\partial}{\partial \dot{q}^i} \tag{12}$$

where

$$B^i \, \frac{\partial}{\partial \dot{q}^i}(\frac{\partial L}{\partial \dot{q}^j}) + \dot{q}^i \, \frac{\partial}{\partial q^i}(\frac{\partial L}{\partial \dot{q}^j}) - \frac{\partial L}{\partial q^j} = 0 \tag{13}$$

Now, if $(q^i(t), \dot{q}^i(t))$ is an integral curve of $\xi_L$, then it satisfies the usual Euler-Lagrange equations

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}^i}) - \frac{\partial L}{\partial q^i} = 0 \tag{14}$$

## Exercise 1: SODEs

A SODE $\xi$ on $TQ$ is a vector field on $TQ$ such that $S(\xi) = \Delta$.

1. Compute the local form of a SODE $\xi$ in bundle coordinates.
2. Prove that an integral curve $\sigma(t)$ of $\xi$ is the tangent lift of its projection $\tau_Q \circ \sigma$ on $Q$, where $\tau_Q : TQ \longrightarrow Q$ is the canonical projection.
3. Prove similar results for a time dependent SODE.

## Exeercise 2: Legendre transformation

1. Prove that the Legendre transformation transport the Euler-Lagrange vector field into the Hamiltonian vector field.
2. How could we define a converse of the Legendre transformation if we start from a Hamiltonian $H$ on $T^*Q$?

## Exercise 3: Geodesic spray

Let $L$ be a Lagrangian on $TQ$ defined by a Riemannian metric $g$ on $Q$, say

$$L(q^i, \dot{q}^i) = (1/2) g_{ij} \dot{q}^i \dot{q}^j$$

where $g_{ij}$ are the components of the metric in a coordinate system $(q^i)$ on $Q$.

1. Prove that the Euler-Lagrange vector field $\xi_L$ is just the geodesic spray.
2. Prove that the Euler-Lagrange equations coincide with the geodesics of the metric.

## Cosymplectic Lagrangian formalism

We will recall here the geometric formalism for time-dependent Lagrangian systems, just to show the differences with the previous contact formalism. In this case, we also have a regular Lagrangian $L : TQ \times \mathbb{R} \longrightarrow \mathbb{R}$, but instead to consider the contact 1-form $\eta_L$ we will consider the cosymplectic structure given by the pair $(\Omega_L, dz)$, where

$$\Omega_L = -d\lambda_L$$

It is esay to check that, indeed, if $L$ is regular then

$$dz \wedge \Omega_L^n \neq 0,$$

and conversely. Again, we have a Reeb vector field

$$\mathcal{R} = \frac{\partial}{\partial z} - W^{ij} \frac{\partial^2 L}{\partial \dot{q}^j \partial z} \frac{\partial}{\partial \dot{q}^i}$$

Consider now the following vector fields determined by means of the vector bundle isomorphism

$$\widetilde{\flat_L} : T(TQ \times \mathbb{R}) \longrightarrow T^*(TQ \times \mathbb{R})$$
$$\widetilde{\flat_L}(v) = i_v \, \Omega_L + dz(v) \, dz$$

say,

1. the gradient vector field

$$\mathrm{grad}\,(E_L) = \widetilde{\sharp_L}(dE_L)$$

2. the Hamiltonian vector field

$$X_{E_L} = \mathcal{E}_L - \mathcal{R}(E_L)\,\mathcal{R}$$

3. and the evolution vector field

$$\mathcal{E}_L = X_{E_L} + \mathcal{R}$$

where $\widetilde{\sharp_L} = (\widetilde{\flat_L})^{-1}$ is the inverse of $\widetilde{\flat_L}$.

The evolution vector field $\mathcal{E}_L$ is locally given by

$$\mathcal{E}_L = \dot{q}^i \frac{\partial}{\partial q^i} + B^i \frac{\partial}{\partial \dot{q}^i} + \frac{\partial}{\partial z} \tag{15}$$

where

$$B^i \frac{\partial}{\partial \dot{q}^i}\left(\frac{\partial L}{\partial \dot{q}^j}\right) + \dot{q}^i \frac{\partial}{\partial q^i}\left(\frac{\partial L}{\partial \dot{q}^j}\right) - \frac{\partial L}{\partial q^j} = 0 \tag{16}$$

Now, if $(q^i(t), \dot{q}^i(t), z(t))$ is an integral curve of $\mathcal{E}_L$ then it satisfies the usual Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \tag{17}$$

since $z = t + constant$.

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## Contact Lagrangian formalism

Let $L : TQ \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Lagrangian function, where $Q$ is a configuration $n$-dimensional manifold. Then, $L = L(q^i, \dot{q}^i, z)$, where $(q^i)$ are coordinates in $Q$, $(q^i, \dot{q}^i)$ are the induced bundle coordinates in $TQ$ and $z$ is a global coordinate in $\mathbb{R}$.

We will assume that $L$ is regular, that is, the Hessian matrix

$$\left( \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right)$$

is regular.

From $L$, and using the canonical endomorphism $S$ on $TQ$ locally defined by

$$S = dq^i \otimes \frac{\partial}{\partial \dot{q}^i}$$

one can construct a 1-form $\lambda_L$ defined by

$$\lambda_L = S^*(dL)$$

where now $S$ and $S^*$ are the natural extension of $S$ and its adjoint operator $S^*$ to $TQ \times \mathbb{R}$.

Therefore, we have

$$\lambda_L = \frac{\partial L}{\partial \dot{q}^i}\, dq^i$$

Now, the 1-form

$$\eta_L = dz - \frac{\partial L}{\partial \dot{q}^i}\, dq^i$$

is a contact form on $TQ \times \mathbb{R}$ if and if $L$ is regular; indeed, if $L$ is regular, then

$$\eta_L \wedge (d\eta_L)^n \neq 0,$$

and conversely. From now on, we always assume that it is the case. The corresponding Reeb vector field is

$$\mathcal{R} = \frac{\partial}{\partial z} - W^{ij} \frac{\partial^2 L}{\partial \dot{q}^j \partial z} \frac{\partial}{\partial \dot{q}^i},$$

where $(W^{ij})$ is the inverse matrix of the Hessian $(W_{ij})$. The energy of the systems is defined by

$$E_L = \Delta(L) - L$$

where $\Delta = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}$ is the Liouville vector field on $TQ$ extended in the usual way to $TQ \times \mathbb{R}$. Therefore,

$$E_L = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L$$

Denote by

$$\bar{\flat}_L : T(TQ \times \mathbb{R}) \longrightarrow T^*(TQ \times \mathbb{R})$$

the vector bundle isomorphism

$$\bar{\flat}_L(v) = i_v(d\eta_L) + (i_v\eta_L)\,\eta_L$$

given by the contact form $\eta_L$ on $TQ \times \mathbb{R}$. We shall denote its inverse by $\bar{\sharp}_L = (\bar{\flat}_L)^{-1}$.
Denote by $\bar{\xi}_L$ the unique vector field defined by the equation

$$\bar{\flat}_L(\bar{\xi}_L) = dE_L - (\mathcal{R}(E_L) + E_L)\,\eta_L \tag{18}$$

A direct computation from eq. (18) shows that $\bar{\xi}_L$ is locally given by

$$\bar{\xi}_L = \dot{q}^i\,\frac{\partial}{\partial q^i} + \mathcal{B}^i\,\frac{\partial}{\partial \dot{q}^i} + L\,\frac{\partial}{\partial z} \tag{19}$$

where the components $\mathcal{B}^i$ satisfy the equation

$$\mathcal{B}^i\,\frac{\partial}{\partial \dot{q}^i}(\frac{\partial L}{\partial \dot{q}^j}) + \dot{q}^i\,\frac{\partial}{\partial q^i}(\frac{\partial L}{\partial \dot{q}^j}) + L\frac{\partial}{\partial z}(\frac{\partial L}{\partial \dot{q}^i}) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z} \tag{20}$$

Then, if $(q^i(t), \dot{q}^i(t), z(t))$ is an integral curve of $\bar{\xi}_L$, and substituting its values in eq. (20) we obtain

$$\ddot{q}^i \frac{\partial}{\partial \dot{q}^i}(\frac{\partial L}{\partial \dot{q}^j}) + \dot{q}^i \frac{\partial}{\partial q^i}(\frac{\partial L}{\partial \dot{q}^j}) + \dot{z}\frac{\partial}{\partial z}(\frac{\partial L}{\partial \dot{q}^i})) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z}$$

which corresponds to the generalized Euler-Lagrange equations considered by G. Herglotz in 1930.

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}^i}) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z} \tag{21}$$

G. Herglotz: Beruhrungstransformationen, Lectures at the University of Gottingen, Gottingen, 1930.

## The Hamiltonian formalism

Let $H : T^*Q \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Hamiltonian function, say $H = H(q^i, p_i, z)$ where $(q^i, p_i, z)$ are bundle coordinates in $H : T^*Q \times \mathbb{R}$. Consider the 1-form

$$\eta_Q = dz - \theta_Q$$

where $\theta_Q$ is the canonical Liouville form on $T^*Q$ and we are considering the usual identifications for a form on $T^*Q$ or $\mathbb{R}$ and its pull-back to $T^*Q \times \mathbb{R}$. In local coordinates, we have

$$\eta_Q = dz - p_i\, dq^i$$

So, $\eta_Q$ is a contact form on $T^*Q \times \mathbb{R}$ and $(q^i, p_i, z)$ are Darboux coordinates. Therefore, we can obtain a Hamiltonian vector field $X_H$ which locally takes the same form that above.

## Legendre transformation

Given a Lagrangian function $L : TQ \times \mathbb{R} \longrightarrow \mathbb{R}$ we can define the Legendre transformation

$$FL : TQ \times \mathbb{R} \longrightarrow T^*Q \times \mathbb{R}$$

given by

$$FL(q^i, \dot{q}^i, z) = (q^i, \hat{p}_i, z)$$

where

$$\hat{p}_i = \frac{\partial L}{\partial \dot{q}^i}$$

A direct computation shows that

$$FL^* \eta_Q = \eta_L$$

and then we have

$$T(FL)(\bar{\xi}_L) = X_H$$

and consequently the generalized Euler-Lagrange (or Herglotz) equations are transformed into the contact Hamilton equations.

## Variational formulation of contact Lagrangian mechanics: Herglotz principle

## Variational formulation of contact Lagrangian mechanics

Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be a Lagrangian function. We will recall the so-called Herglotz's principle, a modification of Hamilton's principle that allows us to obtain Herglotz's equations, sometimes called generalized Euler-Lagrange equations.

Fix $q_1, q_2 \in Q$ and an interval $[a, b] \subset \mathbb{R}$. We denote by $\Omega(q_1, q_2, [a, b]) \subseteq (\mathcal{C}^\infty([a, b] \to Q))$ the space of smooth curves $\xi$ such that $\xi(a) = q_1$ and $\xi(b) = q_2$. This space has the structure of an infinite dimensional smooth manifold whose tangent space at $\xi$ is given by the set of vector fields over $\xi$ that vanish at the endpoints, that is,

$$\begin{aligned} T_\xi \Omega(q_1, q_2, [a, b]) = \{ & v_\xi \in \mathcal{C}^\infty([a, b] \to TQ) \mid \\ & \tau_Q \circ v_\xi = \xi,\ v_\xi(a) = 0,\ v_\xi(b) = 0 \}. \end{aligned} \tag{22}$$

We will consider the following maps. Fix $c \in \mathbb{R}$. Let

$$\mathcal{Z} : \Omega(q_1, q_2, [a, b]) \to \mathcal{C}^{\infty}([a, b] \to \mathbb{R}) \tag{23}$$

be the operator that assigns to each curve $\xi$ the curve $\mathcal{Z}(\xi)$ that solves the following ODE:

$$\frac{d\mathcal{Z}(\xi)(t)}{dt} = L(\xi(t), \dot{\xi}(t), \mathcal{Z}(\xi)(t)), \quad \mathcal{Z}(\xi)(a) = c. \tag{24}$$

Now we define the *action functional* as the map which assigns to each curve the solution to the previous ODE evaluated at the endpoint:

$$\begin{aligned} \mathcal{A} : \Omega(q_1, q_2, [a, b]) &\to \mathbb{R}, \\ \xi &\mapsto \mathcal{Z}(\xi)(b), \end{aligned} \tag{25}$$

that is, $\mathcal{A} = ev_b \mathcal{Z}$, where $ev_b : \zeta \mapsto \zeta(b)$ is the evaluation map at $b$.

**Theorem**(Contact variational principle) Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be a Lagrangian function and let $\xi \in \Omega(q_1, q_2, [a, b])$ be a curve in $Q$. Then, $(\xi, \dot{\xi}, \mathcal{Z}(\xi))$ satisfies the Herglotz's equations if and only if $\xi$ is a critical point of $\mathcal{A}$. This theorem generalizes Hamilton's Variational Principle.

In the case that the Lagrangian is independent of the $\mathbb{R}$ coordinate (i.e., $L(x, y, z) = \hat{L}(x, y)$) the contact Lagrange equations reduce to the usual Euler-Lagrange equations. In this situation, we can integrate the ODE of (25) and we get

$$\mathcal{A}(\xi) = \int_a^b \hat{L}(\xi(t), \dot{\xi}(t))t + \frac{c}{b-a}, \tag{26}$$

that is, the usual Euler-Lagrange action up to a constant.

## Exercise 4: Contact Hamiltonian vector fields

For a Hamiltonian function $H$ on a contact manifold $(M, \eta)$ the Hamiltonian vector field $X_H$ is defined by

$$\bar{\flat}(X_H) = dH - (\mathcal{R}(H) + H)\,\eta$$

Check that this formula is equivalent to these two conditions:

$$\eta(X_H) = -H \ , \ \mathcal{L}_{X_H}\eta = -(\mathcal{R}(H) + H)$$

# 2: A unified geometric framework

## Contact manifolds and Jacobi manifolds

A Jacobi manifold is a triple $(M, \Lambda, E)$, where $\Lambda$ is a bivector field (a skew-symmetric contravariant 2-tensor field) and $E \in \mathfrak{X}(M)$ is a vector field, so that the following identities are satisfied:

$$[\Lambda, \Lambda] = 2E \wedge \Lambda\,, \ \mathcal{L}_E \Lambda = [E, \Lambda] = 0,$$

where $[\cdot, \cdot]$ is the SchoutenNijenhuis bracket.
Given a Jacobi manifold $(M, \Lambda, E)$, we define the *Jacobi bracket*:

$$\begin{aligned} \{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) &\mapsto \mathbb{R}, \\ (f, g) &\mapsto \{f, g\}, \end{aligned}$$

where

$$\{f, g\} = \Lambda(df, dg) + fE(g) - gE(f).$$

This bracket is bilinear, antisymmetric, and satisfies the Jacobi identity. Furthermore it fulfills the weak Leibniz rule:

$$supp(\{f, g\}) \subseteq supp(f) \cap supp(g).$$

That is, $(C^\infty(M), \{\cdot, \cdot\})$ is a local Lie algebra in the sense of Kirillov. Conversely, given a local Lie algebra $(C^\infty(M), \{\cdot, \cdot\})$, we can find a Jacobi structure on $M$ such that the Jacobi bracket coincides with the algebra bracket.
The weak Leibniz rule is equivalent to this identity:

$$\{f, gh\} = g\{f, h\} + h\{f, g\} + ghE(f)$$

Given a contact manifold $(M, \eta)$ we can define a Jacobi structure $(M, \Lambda, E)$ by

$$\Lambda(\alpha, \beta) = -d\eta(\overline{\sharp}\alpha, \overline{\sharp}\beta), \quad E = -\mathcal{R},$$

where $\overline{\sharp} = \overline{\flat}^{-1}$.

## Poisson manifolds

One important particular case of Jacobi manifolds are Poisson manifolds (when $E = 0$). The corresponding Poisson bracket satisfies the following Leibniz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

Examples of Poisson manifolds are symplectic and cosymplectic manifolds.

## Cosymplectic manifold

A cosymplectic manifold is given by a triple $(M, \Omega, \eta)$ where $M$ is a $(2n+1)$-dimensional manifold, $\Omega$ is an exact 2-form and $\eta$ is an exact 1-form.
We consider the isomorfism

$$\begin{array}{rcl} \tilde{\flat} : TM & \to & T_x^*M \\ X & \mapsto & i_X\Omega + \eta(X)\eta. \end{array}$$

If we denote its inverse by $s\tilde{h}arp = \tilde{\flat}^{-1}$, then

$$\Lambda(\alpha, \beta) = \Omega(\tilde{\sharp}\alpha, \tilde{\sharp}\beta),$$

is a Poisson tensor on $M$.

## Locally conformal symplectic manifolds

An almost symplectic manifold is said to be *locally conformally symplectic* if for each point $x \in M$ there is an open neighborhood $U$ such that $d(e^{\sigma}\Omega) = 0$, for $\sigma : U \to \mathbb{R}$, so $(U, e^{\sigma}\Omega)$ is a symplectic manifold. If $U = M$, then it is said to be globally conformally symplectic.
One can see that these local 1-forms $d\sigma$ defines a closed 1-form $\theta$ such that

$$d\Omega = \theta \wedge \Omega.$$

The one-form $\theta$ is called the *Lee one-form*. Locally conformally symplectic manifolds (L.C.S.) with Lee form $\theta = 0$ are symplectic manifolds. We define a bivector $\Lambda$ on $M$ and a vector field $E$ given by

$$\Lambda(\alpha, \beta) = \Omega(\flat^{-1}(\alpha), \flat^{-1}(\beta)) = \Omega(\sharp(\alpha), \sharp(\beta)), \quad E = \flat^{-1}(\theta)$$

with $\alpha, \beta \in \Omega^1(M)$ and $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ is the isomorphism of $C^{\infty}(M)$ modules defined by $\flat(X) = \iota_X \Omega$. Here $\sharp = \flat^{-1}$. The vector field $E$ satisfies $\iota_E \theta = 0$ and $\mathcal{L}_E \Omega = 0, \mathcal{L}_E \theta = 0$. Then, $(M, \Lambda, E)$ is an even dimensional Jacobi manifold.

## Jacobi manifolds (continued)

Let $(M, \Lambda, E)$ be a Jacobi manifold. We define the following morphism of vector bundles:

$$\begin{aligned} \sharp_\Lambda : TM^* &\to TM \\ \alpha &\mapsto \Lambda(\alpha, \cdot), \end{aligned}$$

which also induces a morphism of $C^\infty(M)$-modules between 1-forms and vector fields.
In the case of a contact manifold, this is given by

$$\sharp_\Lambda \alpha = \sharp\alpha - \alpha(\mathcal{R})\mathcal{R},$$

since

$$\eta(\sharp_\Lambda \alpha) = \alpha(\mathcal{R})$$

for any 1-form $\alpha$.
For a contact manifold, $\sharp_\Lambda$ is not an isomorfism. In fact, $\ker \sharp_\Lambda = < \eta >$ and $\mathrm{Im}\, \sharp_\Lambda = \mathcal{H}$.

Vector fields associated with functions $f$ on the algebra of smooth functions $C^\infty(M)$ are defined as

$$X_f = \sharp_\Lambda(df) + fE,$$

The *characteristic distribution* $\mathcal{C}$ of $(M, \Lambda, E)$ is generated by the values of all the vector fields $X_f$. This characteristic distribution $\mathcal{C}$ is defined in terms of $\Lambda$ and $E$ as follows

$$\mathcal{C}_p = \sharp_{\Lambda_p}(T_p^*M) + < E_p >, \quad \forall p \in M$$

where $\sharp_p : T_p^*M \to T_pM$ is the restriction of $\sharp_\Lambda$ to $T_p^*M$ for every $p \in M$. Then, $\mathcal{C}_p = \mathcal{C} \cap T_pM$ is the vector subspace of $T_pM$ generated by $E_p$ and the image of the linear mapping $\sharp_p$.
The distribution is said to be *transitive* if the characteristic distribution is the whole tangent bundle $TM$.

## Distributions and integrability

**Definitions**

- A **singular distribution** $D$ on a manifold $M$ is the assignement to each point $x$ in $M$ a vector subspace $D_x$ of $T_xM$. If the dimension of $D_x$ is constant, then $D$ is called a regular distribution.
- $D$ is called **smooth** if for any point $x \in M$ and any tangent vector $v_x \in D_x$, there is a vector field $X$ on a neighborhood $U$ of $x$ which is tangent to $D$ at any point of $U$ and such that $X(x) = v_x$.
- A distribution $D$ is **invariant** by a vector field $X$ if its flow preserves the distribution.
- A distribution $D$ is **generated** by a family $C$ of vector fields if any $D_x$ is generated by the values of the vector fields of $C$ at $x$.
- $D$ is called **integrable** if for any point $x \in M$ there is a maximal integral submanifold passing through $x$.

**Theorem** (Stefan-Sussmann) Let $D$ be a smooth distribution on $M$. Then the following statements are equivalent:

1. $D$ is integrable.
2. $D$ is generated by a family $C$ of smooth vector fields and invariant by $C$.

## Structure theorem for Jacobi manifolds

**Theorem**

*The characteristic distribution of a Jacobi manifold $(M, \Lambda, E)$ is completely integrable in the sense of Stefan–Sussmann, thus $M$ defines a foliation whose leaves are not necessarily of the same dimension, and it is called the characteristic foliation. Each leaf has a unique transitive Jacobi structure such that its canonical injection into $M$ is a Jacobi map (that is, it preserves the Jacobi brackets). Each can be*

1. *A locally conformally symplectic (or a symplectic) manifold if the dimension is even.*
2. *A manifold equipped with a contact one-form if its dimension is odd.*

## Exercise 5: Characteristic distribution

1. Compute the characteristic distribution in the cases of symplectic, cosymplectic, locally conformal symplectic and contact manifolds.
2. Prove that the characteristic distribution is invariant (in the sense of singular distributions) in the case of contact manifolds.

## Exercise 6: Lagrangian submanifolds and dynamics

Let $(M, \omega, H)$ be a Hamiltonian system and $X_H$ the corresponding Hamiltonian vector field. If $\omega^C$ is the tangent or complete lift of $\omega$ to $TM$, prove:

1. $\omega^C$ is a symplectic form on $TM$
2. $X_H(M) \subset TM$ is a Lagrangian submanifold.

# 3: Dynamics and submanifolds

## Submanifolds of a contact manifold

As in the case of symplectic manifolds, we can consider several interesting types of submanifolds of a contact manifold $(M, \eta)$. To define them, we will use the following notion of *complement* for contact structures:
Let $(M, \eta)$ be a contact manifold and $x \in M$. Let $\Delta_x \subset T_xM$ be a linear subspace. We define the *contact complement* of $\Delta_x$

$$\Delta_x{}^{\perp_\Lambda} = \sharp_\Lambda(\Delta_x{}^o),$$

where $\Delta_x{}^o = \{\alpha_x \in T_x^*M \mid \alpha_x(\Delta_x) = 0\}$ is the annihilator.
We extend this definition for distributions $\Delta \subseteq TM$ by taking the complement pointwise in each tangent space.

Let $N \subseteq M$ be a submanifold. We say that $N$ is:

- *Isotropic* if $TN \subseteq TN^{\perp_\lambda}$.
- *Coisotropic* if $TN \supseteq TN^{\perp_\lambda}$.
- *Legendrian* if $TN = TN^{\perp_\lambda}$.

Let $(M, \eta)$ be a contact manifold of dimension $2n+1$.

One can easily prove the following characterization of a Legendrian submanifold:

**A submanifold $N$ of $M$ is Legendrian if and only if it is a maximal integral manifold of $\ker \eta$ (and then it has dimension $n$).**

## Exercise 7: Characterization of Legendrian submanifolds

Prove this characterization of a Legendre submanifolds:

A submanifold $N$ of $M$ is Legendrian if and only if it is a maximal integral manifold of $\ker\eta$ (and then it has dimension $n$).

**Proposition** A section $\gamma : Q \longrightarrow T^*Q \times \mathbb{R}$ of the canonical projection $T^*Q \times \mathbb{R} \longrightarrow Q$ is a Legendre submanifold of $(T^*Q \times \mathbb{R}, \eta_Q)$ if and only if $\gamma$ is locally the 1-jet of a function $f : Q \longrightarrow \mathbb{R}$.

The above result is the natural extension of the well-known fact that a section $\sigma$ of the cotangent bundle $\pi_Q : T^*Q \longrightarrow Q$ is a Lagrangian submanifold with respect to the canonical symplectic structure $\omega_Q = -d\theta_Q$ on $T^*Q$ if and only if $\sigma$ is a closed 1-form (and hence, locally exact).

## Coisotropic reduction in contact geometry

We will present a result of reduction in the context of contact geometry, which is analogous to the well-known coisotropic reduction in symplectic geometry.

First we note that the horizontal distribution $(\mathcal{H}, d\eta)$ is symplectic. Let be $\Delta \subseteq H$. We denote by $\perp_\eta$ the symplectic orthogonal component

$$\Delta^{\perp_{d\eta}} = \{v \in TM \mid d\eta(v, \Delta) = 0\},$$

We remark that $\mathcal{R} \in \Delta^{\perp_{d\eta}}$ for any distribution $\Delta$. There is a simple relationship between both notions of orthogonal complement:

Let $\Delta \subseteq TM$ be a distribution. Then

$$\Delta^{\perp_\Lambda} = \Delta^{\perp_{d\eta}} \cap \mathcal{H}.$$

We have the following posibilities regarding the relative position of a distribution $\Delta$ in a contact manifold and the vertical and horizontal distributions

**Definition**

Let $\Delta \subseteq TM$ be a distribution of rank $k$. We say that a point $x \in M$ is

1. *Horizontal* if $\Delta_x = \Delta_x \cap \mathcal{H}_x$.
2. *Vertical* if $\Delta_x = (\Delta_x \cap \mathcal{H}_x) \oplus < \mathcal{R}_x >$.
3. *Oblique* if $\Delta_x = (\Delta_x \cap \mathcal{H}_x) \oplus < \mathcal{R}_x + v_x >$, con $v_x \in H_x \setminus \Delta_x$.

If $x$ is horizontal, then $\dim \Delta^{\perp_\Lambda} = 2n - k$. Otherwise, $\dim \Delta^{\perp_\Lambda} = 2n + 1 - k$.

## Characteristic distribution

Given a coisotropic submanifold $\iota : N \longrightarrow M$, we define

$$\eta_0 = \iota^*\eta = \eta TN$$
$$d\eta_0 = \iota^*(d\eta) = d(\iota^*\eta)$$

We call characteristic distribution of $N$ to

$$TN^{\perp_\wedge} = \ker(\eta_0) \cap \ker(d\eta_0).$$

## Contact Hamiltonian systems

Given a smooth function $H$ on a contact manifold $(M, \eta)$, we define its *Hamiltonian vector field* as

$$X_H = \sharp_\Lambda(dH) - H\mathcal{R},$$

or equivalently,

$$\flat(X_H) = dH - (\mathcal{R}(H) + H)\eta.$$

In Darboux coordinates, we have

$$X_H = \frac{\partial H}{\partial y_i}\frac{\partial}{\partial x^i} - (\frac{\partial H}{\partial x^i} + y_i\frac{\partial H}{\partial z})\frac{\partial}{\partial y_i} + (y_i\frac{\partial H}{\partial y_i} - H)\frac{\partial}{\partial z}$$

A *contact Hamiltonian* system is a triple $(M, \eta, H)$, where $(M, \eta)$ is a contact manifold and $H$ is a smooth real function on $M$.

One can easily shows that

$$\mathcal{L}_{X_H} H = -\mathcal{R}(H)H$$

which shows that the system does not preserve the energy.

Let $(M, \eta, H)$ a Hamiltonian contact system with Reeb vector field $\mathcal{R}$ and Hamiltonian dynamics $X_H$. Assume that $M$ has dimension $2n+1$.
A direct computation shows that

$$\begin{aligned}
&\mathcal{L}_{X_H}\eta = -\mathcal{R}(H)\eta \\
&\mathcal{L}_{X_H}d\eta = -d(\mathcal{R}(H))\eta - \mathcal{R}(H)d\eta \\
&\mathcal{L}_{X_H}(\eta \wedge d\eta) = -2\mathcal{R}(H)\eta \wedge d\eta \\
&\mathcal{L}_{X_H}(\eta \wedge (d\eta)^2) = -3\mathcal{R}(H)\eta \wedge (d\eta)^2
\end{aligned}$$

and by induction one can prove that

$$\mathcal{L}_{X_H}(\eta \wedge (d\eta)^n) = -(n+1)\,\mathcal{R}(H)\eta \wedge (d\eta)^n$$

This prove that the contact volume is not preserved.
However,

$$\Omega = H^{-(n+1)}\eta \wedge (d\eta)^n$$

is preserved.

## Morphisms between contact manifolds

A diffeomorphism between two contact manifolds $F : (M, \eta) \to (N, \xi)$ is a *contactomorphism* if

$$F^*\xi = \eta.$$

A diffeomorphism $F : (M, \eta) \to (N, \xi)$ is a *conformal contactomorphism* if there exist a nowhere zero function $f \in C^\infty(M)$ such that

$$F^*\xi = f\eta.$$

A vector field $X \in \mathfrak{X}(M)$ is a *infinitesimal contactomorphism* (respectively *infinitesimal conformal contactomorphism*) if its flow $\phi_t$ consists of contactomorphisms (resp. *conformal contactomorphisms*). A vector field $X$ is an infinitesimal contactomorphism if and only if

$$\mathcal{L}_X\eta = 0.$$

$X$ is an infinitesimal conformal contactomorphism if and only if there exists $g \in C^\infty(M)$ such that

$$\mathcal{L}_X\eta = g\eta.$$

In this case, we say that $(g, X)$ is an *infinitesimal conformal contactomorphism*.

Next, we will investigate the relationship between Hamiltonian vector fields and Legendrian submanifolds.

**Theorem (Contactification of the tangent bundle)**

*Let $(M, \eta)$ be a contact manifold. Let $\bar{\eta}$ be a one form on $TM \times \mathbb{R}$ such that*

$$\bar{\eta} = \eta^C + t\eta^V,$$

*where $t$ is the usual coordinate on $\mathbb{R}$ and $\eta^C$ and $\eta^V$ are the complete and vertical lifts of $\eta$ to $TM$. Then, $(TM \times \mathbb{R}, \bar{\eta})$ is a contact manifold with Reeb vector field $\bar{\mathcal{R}} = \mathcal{R}^V$.*

**Theorem**

*Let $(M, \eta)$ be a contact manifold, and let $X \in \mathfrak{X}(M)$, $f \in C^\infty(M)$. We denote*

$$\begin{aligned} X \times f : M &\to TM \times \mathbb{R} \\ p &\mapsto (X_p, f(p)), \end{aligned}$$

*Then $(f, X)$ is an infinitesimal conformal contactomorphism if and only if $(X \times f) \subseteq (TM \times \mathbb{R}, \bar{\eta})$ is a Legendrian submanifold.*

This result states that the image of vector field $X_H$, suitably included in the contactified tangent bundle, is a Legendrian submanifold. In this sense, Hamiltonian vector fields are particular cases of Legendrian submanifolds.

**Theorem**

*Let $(M, \eta, H)$ be a contact Hamiltonian system. Then*

$$(X_H \times (\mathcal{R}(H))) \subseteq (TM \times \mathbb{R}, \bar{\eta})$$

*is a Legendrian submanifold.*

The result follows since

$$\mathcal{L}_{X_H} \eta = -\mathcal{R}(H)\eta$$

**Theorem (Coisotropic reduction in contact manifolds)**

*Let $\iota : N \longrightarrow M$ be a coisotropic submanifold. Then $TN^{\perp_\Lambda}$ is involutive. If the quotient $\tilde{N} = TN/TN^{\perp_\Lambda}$ is a manifold and $N$ does not have horizontal points, let $\pi : N \to \tilde{N}$ be the projection. Then there exists a unique 1-form $\tilde{\eta}$ on $\tilde{N}$ such that $\tilde{\eta} = \pi_*(\eta)$ and $(N, \tilde{\eta})$ is a contact manifold.*
*Furthermore, if $N$ consists only of vertical points, then $\tilde{\mathcal{R}} = \pi^*\mathcal{R}$ is well defined and is the correspondig Reeb vector field.*

The following theorem is very related to a similar result in

A.G. Tortorella: Rigidity of Integral Coisotropic Submanifolds of Contact Manifolds. *Letters in Mathematical Physics* **108** 3 (2018), 883-896.

Indeed, this result provides a coisotropic reduction theorem for *regular coisotropic submanifolds* which coincides with our notion of *coisotropic submanifolds without horizontal points*, but it is used in a slightly differente context.
There is another, non-equivalent, widespread definition of contact manifold. Some authors define contact manifolds $(M, \xi)$ as odd-dimensional manifolds $M$ with a *contact distribution* $\xi$, that is, a maximally non-integrable codimension 1 distribution. By the Frobenius theorem, this means that $\xi$ is given locally as the kernel of a contact form $\eta$. Of course, every contact manifold $(M, \eta)$ is a contact manifold in this sense by taking $\xi = \ker \eta$. Conversely, a contact distribution $\xi$ is globally the kernel of contact form if and only if $\xi$ is co-orientable.

## Corollary

*With the notations from previous theorem, assume that $L \subseteq M$ is Legendrian, $N$ does not have horizontal points, and $N$ and $L$ have clean intersection (that is, $N \cap L$ is a submanifold and $T(N \cap L) = TN \cap TL$). Then $\tilde{L} = \pi(L) \subseteq \tilde{N}$ is Legendrian.*

# 4: Symmetries and reduction

## Infinitesimal symmetries and Noether Theorem

Noether theorem is one of the most relevant results relating symmetry groups of a Lagrangian system and conserved quantities of the corresponding Euler-Lagrange equations. In the simplest view, the existence of a *cyclic* coordinate implies the conservation of the corresponding momentum. Indeed, if $L = L(q^i, \dot{q})$ does not depend on the coordinate $q^j$, then, using the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^j}\right) - \frac{\partial L}{\partial q^j} = 0, \tag{27}$$

we deduce that

$$\dot{p}_j = \frac{\partial L}{\partial \dot{q}^j} = 0. \tag{28}$$

Noether theorem can be described on a geometric framework.
Now, $L$ is a function on the tangent bundle $TQ$ of the configuration manifold $Q$ and $X$ be a vector field on $Q$. Denote by $X^V$ and $X^C$ the vertical and complete lifts of $X$ to $TQ$. Then:
**Theorem** (Noether Theorem)
$X^C(L) = 0$ if and only if $X^V(L)$ is a conserved quantity.

In contact Lagrangian dynamics, the generalized Euler-Lagrange equations look as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^j}\right) - \frac{\partial L}{\partial q^j} = \frac{\partial L}{\partial \dot{q}^j}\frac{\partial L}{\partial z}, \tag{29}$$

and if we insist to proceed as in the symplectic case, we would have

$$\dot{p}_j = \frac{\partial L}{\partial z} p_j$$

Therefore,

$$p_j = exp \int \frac{\partial L}{\partial z}$$

## Infinitesimal symmetries in contact Hamiltonian systems

Let $(M, \eta, H)$ a contact Hamiltonian systems with Reeb vector field $\mathcal{R}$. The Jacobi bracket of two functions $f, g \in C^\infty(M)$ is given by

$$\{f, g\} = \Lambda(df, dg) - f\mathcal{R}(g) + g\mathcal{R}(f)$$

where $(\Lambda, E = -\mathcal{R})$ is the associated Jacobi structure to $(M, \eta)$. Let $X_f$ the Hamiltonian vector field defined by a function $f$.

These two lemmas are essential for our purposes:

**Lemma 1** We have

$$\{f, g\} = X_f(g) + g\mathcal{R}(f)$$

This implies that

$$X_H(f) = \{H, f\} - \mathcal{R}(H)\, f$$

so that an observable $f$ dissipates at the same rate that the Hamiltonian if and only if $f$ and $H$ commute (and in that case, $\frac{f}{H}$ is a conserved quantity.

**Lemma 2** We have

$$\{f, g\} = -\eta([X_f, X_g])$$

**Proposition** Let $X$ be a vector field on $M$ such that $\eta(X) = -f$. Then

$$\{H, f\} = -\eta([X_H, X]) = (\mathcal{L}_X \eta)(X_H) + X(H)$$

**Proof**: If $\eta(X) = -f$, then $\eta(X - X_f) = 0$, so that $X - X_f$ is in the kernel of $\eta$.
Since

$$\mathcal{L}_X \eta = -\mathcal{R}(H)\eta$$

we deduce that

$$(\mathcal{L}_X \eta)(X_f) = (\mathcal{L}_X \eta)(X)$$

Therefore

$$\begin{aligned}
\{H, f\} &= -\eta([X_H, X_f])(\text{using Lemma 2}) \\
&= (\mathcal{L}_{X_H} \eta)(X_f) - X_H(\eta(X_f))(\text{by using the Cartan formula}) \\
&= (\mathcal{L}_{X_H}\eta)(X) - X_H(\eta(X)) \\
&= -\eta([X_H, X])(\text{again by using the Cartan formula})
\end{aligned}$$

From the second equality, we have

$$\begin{aligned}
-\eta([X_H, X]) &= (\mathcal{L}_X \eta)(X_H) - X(\eta(X_H))(\text{by using the Cartan formula}) \\
&= (\mathcal{L}_X\eta)(X_H) + X(H)
\end{aligned}$$

The above Proposition suggests us to introduce the following definition. **Definition**. A vector field $X$ on $M$ such that

$$\eta([X_H, X]) = 0$$

will be called a *dynamical symmetry* for $(M, \eta, H)$.

Using the above Lemmas and the previous Proposition, the following result is immediate.

**Theorem** Let $X$ be a vector field on $M$. Then $X$ is a dynamical symmetry for $(M, \eta, H)$ if and only if $\eta(X)$ commutes with $H$.

## Infinitesimal symmetries in contact Lagrangian systems

In this case, we will take benefit fom the bundle structure of $TQ \times \mathbb{R}$. For a vector field $X = X^i \frac{\partial}{\partial q^i}$ on $Q$, we will denote its vertical and complete lifts to $TQ$ (and with the natural extension to $TQ \times \mathbb{R}$ by

$$X^V = X^i \frac{\partial}{\partial \dot{q}^i}$$

$$X^C = \frac{\partial}{\partial q^i} + \dot{q}^j \frac{\partial X^i}{\partial q^j} \frac{\partial}{\partial \dot{q}^i}$$

Next, let $Y$ be a vector field on $Q \times \mathbb{R}$. If

$$Y = Y^i \frac{\partial}{\partial q^i} + \mathcal{Z} \frac{\partial}{\partial z}$$

then its complete lift to $T(Q \times \mathbb{R})$ is

$$\begin{aligned} Y^C &= Y^i \frac{\partial}{\partial q^i} + \mathcal{Z} \frac{\partial}{\partial z} + \dot{q}^j \frac{\partial Y^i}{\partial q^j} \frac{\partial}{\partial \dot{q}^i} \\ &\quad + \dot{q}^j \frac{\partial \mathcal{Z}}{\partial q^j} \frac{\partial}{\partial \dot{z}} + \dot{z} \frac{\partial Y^i}{\partial z} \frac{\partial}{\partial \dot{q}^i} + \dot{z} \frac{\partial \mathcal{Z}}{\partial z} \frac{\partial}{\partial \dot{z}} \end{aligned}$$

Here $(z, \dot{z})$ are the bundle coordinates in $T\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$.

Since we are restricted to the submanifold $TQ \times \mathbb{R}$ of $T(Q \times \mathbb{R})$ we consider only such vector fields $Y$ on $Q \times \mathbb{R}$ such that its complete lift to $T(Q \times \mathbb{R})$ be tangent to $TQ \times \mathbb{R}$. This just happens when

$$\frac{\partial \mathcal{Z}}{\partial q^i} = 0$$

that is, $Z$ does not depend on the positions $q$. The restriction of such $Y^C$ to $TQ \times \mathbb{R}$ will be denoted by

$$\bar{Y}^C = Y^i \frac{\partial}{\partial q^i} + \mathcal{Z} \frac{\partial}{\partial z} + \dot{q}^j \frac{\partial Y^i}{\partial q^j} \frac{\partial}{\partial \dot{q}^i}$$

In such a case, we will denote by $\bar{Y}^V$ the vertical lift of the projection of $Y$ to $Q$, say

$$\bar{Y}^V = Y^i \frac{\partial}{\partial \dot{q}^i}$$

which is obviouly tangent to $TQ \times \mathbb{R}$

Next, we shall consider a contact Lagrangian system given by a Lagrangian $L : TQ \times \mathbb{R} \longrightarrow \mathbb{R}$. The corresponding contact Hamiltonian system is $(TQ \times \mathbb{R}, \eta_L, E_L)$ with the obvious notations. $calR_L$ is the Reeb vector field and $\xi_L$ the Euler-Lagrange vector field.

**Definition** A vector field $X$ on $Q$ is called an *infinitesimal symmetry* of $L$ if $X^C(L) = 0$.

**Theorem** A vector field $X$ on $Q$ is an infinitesimal symmetry of $L$ if and only if the function

$$f = X^V(L)$$

commutes with the energy, that is,

$$\xi_L(f) = -\mathcal{R}_L(E_L)f = \frac{\partial L}{\partial z} f$$

Notice that if $X$ is an infinitesimal symmetry of $L$, then $X^C$ is the Hamiltonian vector field of $X^V(L)$, say

$$X^C = X_{X^V(L)}$$

The above definition can be slightly extended as follows

**Definition** Let $Y$ a vector field on $Q \times \mathbb{R}$ such that $Y^C$ is tangent to $TQ \times \mathbb{R}$. Then $Y$ is called a *generalized infinitesimal symmetry* of $L$ if

$$\bar{Y}^C(L) = -\mathcal{R}_L(f)L$$

where

$$f = \bar{Y}(L) - \mathcal{Z}$$

and $\mathcal{Z}$ is the $z$-component of $Y$.

**Theorem** Let $Y$ be a *generalized infinitesimal symmetry* of $L$. Then

$$f = \bar{Y}^V(L) - \mathcal{Z}$$

commutes with $E_L$, and, conversely, in that case, $Y$ is a generalized infinitesimal symmetry of $L$.

More types of infinitesimal symmetries

**Definition (Cartan and Noether symmetries)** A vector field $\tilde{Y}$ on $TQ \times \mathbb{R}$ is called a *Cartan symmetry* if

$$\mathcal{L}_{\tilde{Y}}\, \eta_L = a\, \eta_L + dg;\, \tilde{Y}(E_L) = aE_L + g\mathcal{R}_L(E_L)$$

for some functions $a, g \in C^\infty(TQ \times \mathbb{R})$.
A vector field $Y$ on $Q \times \mathbb{R}$ such that $Y^C$ is tangent to $TQ \times R$ is called a *Noether symmetry* if $\bar{Y}^C$ is a Cartan symmetry.

**Theorem**
(1) If $Y$ is a Noether symmetry such that

$$\bar{Y}^C(L) = g^C$$

then

$$f = \bar{Y}^V(L) - g^V$$

commutes with $E_L$.
(2) If $\tilde{Y}$ is a Cartan symmetry such that

$$\mathcal{L}_{\tilde{Y}}\, \eta_L = dg$$

then

$$\eta_Y(\tilde{Y}) - g$$

commutes with $E_L$.

**Definition (Lie symmetries)**. A vector field $Y$ on $Q \times R$ such that $Y^C$ is tangent to $TQ \times R$ and $\bar{Y}^C$ is a dynamical symmetry will be called a *Lie symmetry*.

**Theorem**. If $Y$ is a Lie symmetry, then

$$-\eta_L(\bar{Y}^C) = \bar{Y}^V(L) - \mathcal{Z}$$

commutes with $E_L$.

## Momentum maps

The moment map is well-known in symplectic geometry. There is a contact analog that we will descibe in the next slides.

### Definition

Let $(M, \eta)$ be a contact manifold and let $G$ be a Lie group acting on $M$ by contactomorphisms. In analogy to the exact symplectic case, we define the moment map $J : M \to \mathfrak{g}^*$ such that

$$J(x)(\xi) = -\eta(\xi_M(x)),$$

where $x \in M$, $\xi \in \mathfrak{g}$ and $\xi_M$ is the the infinitesimal generator of the action corresponding to $\xi$.

We have

$$X_{\hat{J}_\xi} = \xi_M$$

where $X_{\hat{J}_\xi}$ is the Hamiltonian vector field corresponding to the function $\hat{J}_\xi(x) =< J(x), \xi >$.
The moment map defined is equivariant under the coadjoint action. That is, we have

$$Ad^*_{g^{-1}} \circ J = J \circ g$$

$g \in G$, $\alpha \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$, where $Ad^* : G \to Aut(\mathfrak{g}^*)$ is the coadjoint representation.

## Proposition

Let $(M, \eta)$ be a contact manifold on which a Lie group $G$ acts by contactomorphisms. Let $\mu \in \mathfrak{g}^*$ be a regular value of the moment map $J$. Then, for all $x \in J^{-1}(\mu)$ we have

$$T_x(G_\mu x) = T_x(Gx) \cap T_x(J^{-1}(\mu)),$$

where $G_\mu = \{g \in G \mid Ad^*_{g^{-1}}\mu = \mu\}$ is the isotropy group of $\mu$ with respect to the coadjoint action.
We also have

$$T_x(J^{-1}(\mu)) = T_x(Gx)^{\perp d\eta}.$$

In particular, if $G = G_\mu$, then $T_x(Gx) \subseteq T_x(J^{-1}(\mu))$ and $T_x(J^{-1}(\mu))$ is coisotropic and consists of vertical points. Furthermore

$$T_x(J^{-1}(\mu))^{\perp_\Lambda} = T_x(Gx)$$

## Contact reduction

Let $(M, \eta)$ be a contact manifold on which a Lie group $G$ acts freely and properly by contactomorphisms and let $J$ be the momentum map. Let $\mu \in \mathfrak{g}$ be a regular value of $J$ which is a fixed point of $G$ under the coadjoint action. Then, $M_\mu = J^{-1}(\mu)/G$ has a unique contact form $\eta_\mu$ such that

$$\pi_\mu^* \eta_\mu = \iota_\mu^* \eta,$$

where $\pi_\mu : J^{-1}(\mu) \to M_\mu$ is the canonical projection and $\iota_\mu : J^{-1}(\mu) \to M$ is the inclusion.
Also the Reeb vector field of the quotient $\mathcal{R}_\mu = \pi_\mu^* \mathcal{R}$ is the projection of the Reeb vector field $\mathcal{R}$ of $(M, \eta)$.

## Dynamics reduction

Let $G$ be a group acting by contactomorphisms on $(M, \eta, H)$ such that $H$ is $G$-invariant. Then, $(M_\mu, \eta_\mu, H_\mu)$ is a Hamiltonian system, where $H_\mu$ is quotient of $H$ by the action of $G$, that is

$$\pi_{\mu *} X_{H|_{J^{-1}(\mu)}} = X_{H_\mu}$$

$$\begin{tikzcd}
M \arrow[dr, "H"] & \\
J^{-1}(\mu) \arrow[u, "\iota_\mu"] \arrow[d, "\pi_\mu"'] & \mathbb{R} \\
M_\mu \arrow[ur, "H_\mu"'] &
\end{tikzcd}$$

### Exercise 9

Construct a natural momentum map for this canonical contact structure $(T^*Q \times R, \eta_Q)$, when $G$ is a Lie group acting on $Q$.

# 5: Hamilton-Jacobi theory

## Classical Hamilton-Jacobi theory (geometric version)

The standard formulation of the Hamilton-Jacobi problem is to find a function $S(t, q^A)$ (called the **principal function**) such that

$$\frac{\partial S}{\partial t} + h(q^A, \frac{\partial S}{\partial q^A}) = 0. \tag{30}$$

If we put $S(t, q^A) = W(q^A) - tE$, where $E$ is a constant, then $W$ satisfies

$$h(q^A, \frac{\partial W}{\partial q^A}) = E; \tag{31}$$

$W$ is called the **characteristic function**.
Equations (30) and (31) are indistinctly referred as the **Hamilton-Jacobi equation**.
R. Abraham, J.E. Marsden: *Foundations of Mechanics* (2nd edition). Benjamin-Cumming, Reading, 1978.

Let $M$ be the configuration manifold, and $T^*M$ its cotangent bundle equipped with the canonical symplectic form

$$\omega_M = dq^A \wedge dp_A$$

where $(q^A)$ are coordinates in $M$ and $(q^A, p_A)$ are the induced ones in $T^*M$.
Let $h : T^*M \longrightarrow \mathbb{R}$ a hamiltonian function and $X_h$ the corresponding hamiltonian vector field:

$$i_{X_h} \omega_M = dh$$

The integral curves of $X_h$, $(q^A(t), p_A(t))$, satisfy the Hamilton equations:

$$\frac{dq^A}{dt} = \frac{\partial h}{\partial p_A}, \ \frac{dp_A}{dt} = -\frac{\partial h}{\partial q^A}$$

Let $\lambda$ be a closed 1-form on $M$, say $d\lambda = 0$; (then, locally $\lambda = dW$).

**Hamilton-Jacobi Theorem**
The following conditions are equivalent:

**(i)** If $\sigma : I \to M$ satisfies the equation

$$\frac{dq^A}{dt} = \frac{\partial h}{\partial p_A}$$

then $\lambda \circ \sigma$ is a solution of the Hamilton equations;

**(ii)** $d(h \circ \lambda) = 0$

Define a vector field on $M$:

$$X_h^\lambda = T\pi_M \circ X_h \circ \lambda$$

$$\begin{tikzcd}
T^*M \arrow[rr, "X_h"] \arrow[d, "\pi_M"] & & T(T^*M) \arrow[d, "T\pi_M"] \\
M \arrow[u, bend left, "\lambda"] \arrow[rr, "X_h^\lambda"] & & TM
\end{tikzcd}$$

The following conditions are equivalent:

**(i)** If $\sigma : I \to M$ satisfies the equation

$$\frac{dq^A}{dt} = \frac{\partial h}{\partial p_A}$$

then $\lambda \circ \sigma$ is a solution of the Hamilton equations;

**(i)'** If $\sigma : I \to M$ is an integral curve of $X_h^\lambda$, then $\lambda \circ \sigma$ is an integral curve of $X_h$;

**(i)"** $X_h$ and $X_h^\lambda$ are $\lambda$-related, i.e.

**Hamilton-Jacobi Theorem** Let $\lambda$ be a closed 1-form on $M$. Then the following conditions are equivalent:

**(i)** $X_h^\lambda$ and $X_h$ are $\lambda$-related;

**(ii)** $d(h \circ \lambda) = 0$

If

$$\lambda = \lambda_A(q)\, dq^A$$

then the Hamilton-Jacobi equation becomes

$$h(q^A, \lambda_A(q^B)) = const.$$

and we recover the classical formulation when

$$\lambda_A = \frac{\partial W}{\partial q^A}$$

## The Hamiltonian–Jacobi theory for contact Hamiltonian systems

Let $H = H(q^i, p_i, z)$ be a Hamiltonian with contact Hamilton equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}\ ,\ \frac{dp_i}{dt} = -(\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial z})\ ,\ \frac{dz}{dt} = p_i \frac{\partial H}{\partial p_i} - H \qquad (32)$$

or

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}\ ,\ \frac{dp_i}{dt} = -(\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial z})\ ,\ \frac{dz}{dt} = p_i \frac{\partial H}{\partial p_i} \qquad (33)$$

Let $S = S(q^i)$ be a function such that

$$H(q^i, \frac{\partial S}{\partial q^i}, S(q^i)) = k \qquad (34)$$

where $k$ is a constant, then the curve

$$(q^i(t), \frac{\partial S}{\partial q^i}(t), S(q^i(t))$$

is a solution of (33), assuming that

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}$$

- What is the geometric version of these equations?
- What kind of submanifolds now play the role of Lagrangian submanifolds?
- In this talk we will give some answers, contained in the following papers:
  - de León, M.; Sardón, C.: Cosymplectic and contact structures for time-dependent and dissipative Hamiltonian systems. *J. Phys. A* 50 (2017), no. 25, 255205, 23 pp.
  - de León, M.; Lainz, M.; Muñiz-Brea, A.: The HamiltonJacobi theory for contactHamiltonian systems. *arXiv preprint arXiv:2103.17017*

Given a contact $2n+1$ dimensional manifold $(M, \eta)$, we can consider the following distributions on $M$, that we will call *vertical* and *horizontal* distribution, respectively:

$$\begin{aligned} \mathcal{H} &= \ker \eta, \\ \mathcal{V} &= \ker d\eta. \end{aligned}$$

We have a Withney sum decomposition

$$TM = \mathcal{H} \oplus \mathcal{V},$$

and, at each point $x \in M$:

$$T_x M = \mathcal{H}_x \oplus \mathcal{V}_x.$$

We will denote by $\pi_{\mathcal{H}}$ and $\pi_{\mathcal{V}}$ the projections onto these subspaces. We notice that $\dim \mathcal{H} = 2n$ and $\dim \mathcal{V} = 1$, and that $(d\eta)_{|\mathcal{H}}$ is non-degenerate and $\mathcal{V}$ is generated by the Reeb vector field $\mathcal{R}$.

## Definition

1. A diffeomorphism between two contact manifolds $F : (M, \eta) \to (N, \xi)$ is a *contactomorphism* if
$$F^*\xi = \eta.$$
2. A diffeomorphism $F : (M, \eta) \to (N, \xi)$ is a *conformal contactomorphism* if there exist a nowhere zero function $f \in C^\infty(M)$ such that
$$F^*\xi = f\eta.$$
3. A vector field $X \in \mathfrak{X}(M)$ is an *infinitesimal contactomorphism* (respectively *infinitesimal conformal contactomorphism*) if its flow $\phi_t$ consists of contactomorphisms (resp. *conformal contactomorphisms*).

Therefore, we have

**Proposition**

1. A vector field $X$ is an infinitesimal contactomorphism if and only if
$$\mathcal{L}_X \eta = 0.$$
2. $X$ is an infinitesimal conformal contactomorphism if and only if there exists $g \in C^\infty(M)$ such that
$$\mathcal{L}_X \eta = g\eta.$$

In this case, we say that $(g, X)$ is an *infinitesimal conformal contactomorphism*.

If $(M, \eta)$ is a $(2n+1)$-dimensional contact manifold and take Darboux coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n, z)$, then

$$\mathcal{V} = < \frac{\partial}{\partial z} > , \ \mathcal{H} = < A_i, B^i >$$

where

$$\begin{aligned} A_i &= \frac{\partial}{\partial q^i} - p_i \frac{\partial}{\partial z} \\ B^i &= \frac{\partial}{\partial p_i}. \end{aligned}$$

$\{A_1, B^1, \ldots, A_n, B^n, \mathcal{R}\}$ and $\{dq^1, dp_1, \ldots, dq^n, dp_n, \eta\}$ are dual basis.
We also have

$$[A_i, B^i] = -\mathcal{R}$$

## Contact manifolds as Jacobi structures

A Jacobi manifold is a triple $(M, \Lambda, E)$, where $\Lambda$ is a bivector field (a skew-symmetric contravariant 2-tensor field) and $E \in \mathfrak{X}(M)$ is a vector field, so that the following identities are satisfied:

$$[\Lambda, \Lambda] = 2E \wedge \Lambda\,, \ \mathcal{L}_E\Lambda = [E, \Lambda] = 0,$$

where $[\cdot, \cdot]$ is the SchoutenNijenhuis bracket. The *Jacobi bracket* on $C^\infty(M)$ is defined by:

$$\{f, g\} = \Lambda(df, dg) + fE(g) - gE(f).$$

This bracket is bilinear, antisymmetric, and satisfies the Jacobi identity. Furthermore, it fulfills the weak Leibniz rule:

$$\mathrm{supp}(\{f, g\}) \subseteq \mathrm{supp}(f) \cap \mathrm{supp}(g).$$

That is, $(C^\infty(M), \{\cdot, \cdot\})$ is a local Lie algebra in the sense of Kirillov. Conversely, given a local Lie algebra $(C^\infty(M), \{\cdot, \cdot\})$, we can find a Jacobi structure on $M$ such that the Jacobi bracket coincides with the algebra bracket.
The weak Leibniz rule is equivalent to this identity:

$$\{f, gh\} = g\{f, h\} + h\{f, g\} + ghE(f)$$

Given a contact manifold $(M, \eta)$ we can define the associated Jacobi structure $(M, \Lambda, E)$ by

$$\Lambda(\alpha, \beta) = -d\eta(\sharp\alpha, \sharp\beta), \quad E = -\mathcal{R},$$

where $\sharp = \bar{\flat}^{-1}$. For an arbitrary function $f$ on $M$ we can prove that the Hamiltonian vector field $X_f$ with respect to the contact structure $\eta$ coincides with the one defined by its associated Jacobi structure, say

$$X_f = \sharp_\Lambda(df) - f\mathcal{R}$$

where $\sharp_\Lambda$ is the vector bundle morphism from tangent covectors to tangent vectors defined by $\Lambda$, i.e.

$$< \sharp_\Lambda(\alpha), \beta >= \Lambda(\alpha, \beta),$$

for all covectors $\alpha$ and $\beta$.

## Submanifolds

As in the case of symplectic manifolds, we can consider several interesting types of submanifolds of a contact manifold $(M, \eta)$. To define them, we will use the following notion of *complement* for contact structures:
Let $(M, \eta)$ be a contact manifold and $x \in M$. Let $\Delta_x \subset T_x M$ be a linear subspace. We define the *contact complement* of $\Delta_x$

$$\Delta_x{}^{\perp_\Lambda} = \sharp_\Lambda(\Delta_x{}^o),$$

where $\Delta_x{}^o = \{\alpha_x \in T_x^* M \mid \alpha_x(\Delta_x) = 0\}$ is the annihilator.
We extend this definition for distributions $\Delta \subseteq TM$ by taking the complement pointwise in each tangent space.
Here, $\Lambda$ is the associated 2-tensor according to the previous section.

## Definition

Let $N \subseteq M$ be a submanifold. We say that $N$ is:

- *Isotropic* if $TN \subseteq TN^{\perp_\Lambda}$.
- *Coisotropic* if $TN \supseteq TN^{\perp_\Lambda}$.
- *Legendrian* or *Legendre* if $TN = TN^{\perp_\Lambda}$.

The coisotropic condition can be written in local coordinates as follows. Let $N \subseteq M$ be a $k$-dimensional manifold given locally by the zero set of functions $\phi_a : U \to \mathbb{R}$, with $a \in \{1, , k\}$.
We have that

$$TN^{\perp_\Lambda} =< Z_a \mid a = 1, \ldots, k >$$

where

$$Z_a = \sharp_\Lambda(d\phi_a)$$

Therefore, $N$ is coisotropic if and only if, $Z_a(\phi_b) = 0$ for all $a, b$.
Notice that

$$Z_a = \left(\frac{\partial \phi_a}{\partial q^i} + p_i \frac{\partial \phi_a}{\partial z}\right) \frac{\partial}{\partial p_i} + \frac{\partial \phi_a}{\partial p_i} \left(\frac{\partial}{\partial q^i} - p_i \frac{\partial}{\partial z}\right). \tag{35}$$

According to (35), we conclude that $N$ is coisotropic if and only if

$$\left(\frac{\partial \phi_a}{\partial q^i} + p_i \frac{\partial \phi_a}{\partial z}\right) \frac{\partial \phi_b}{\partial p_i} + \frac{\partial \phi_a}{\partial p_i} \left(\frac{\partial \phi_b}{\partial q^i} - p_i \frac{\partial \phi_b}{\partial z}\right) = 0. \tag{36}$$

Using the above results, one can easily prove the following characterization of a Legendrian submanifold.

**Proposition** Let $(M, \eta)$ be a contact manifold of dimension $2n+1$. A submanifold $N$ of $M$ is Legendrian if and only if it is a maximal integral manifold of $\ker \eta$ (and then it has dimension $n$).

Consider a function $f : Q \longrightarrow \mathbb{R}$ and let $\eta_Q = dz - \rho^* \theta_Q$ the canonical contact structure on $T^*Q \times \mathbb{R}$. Here $\rho : T^*Q \times \mathbb{R} \longrightarrow T^*Q$ is the canonical projection, and $\theta_Q$ is the canonical Liouville form on $T^*Q$. In bundle coordinates $(q^i, p_i, z)$, we have

$$\eta_Q = dz - p_i \, dq^i$$

so that $(q^i, p_i, z)$ are Darboux coordinates.

We denote by $j^1 f : Q \longrightarrow T^*Q \times \mathbb{R}$ the 1-jet of $f$, say

$$j^1 f(q^i) = \left( q^i, \frac{\partial f}{\partial q^i}, f(q^i) \right)$$

Then, one immediately checks that $j^1 f(Q)$ is a Legendrian submanifold of $(T^*Q \times \mathbb{R}, \eta_Q)$. Moreover, we have

**Proposition** A section $\gamma : Q \longrightarrow T^*Q \times \mathbb{R}$ of the canonical projection $T^*Q \times \mathbb{R} \longrightarrow Q$ is a Legendrian submanifold of $(T^*Q \times \mathbb{R}, \eta_Q)$ if and only if $\gamma$ is locally the 1-jet of a function $f : Q \longrightarrow \mathbb{R}$.

The above result is the natural extension of the well-known fact that a section $\sigma$ of the cotangent bundle $\pi_Q : T^*Q \longrightarrow Q$ is a Lagrangian submanifold with respect to the canonical symplectic structure $\omega_Q = -d\theta_Q$ on $T^*Q$ if and only if $\sigma$ is a closed 1-form (and hence, locally exact).

## The Hamilton-Jacobi equations for a Hamiltonian vector field

We consider the extended phase space $T^*Q \times \mathbb{R}$, and a Hamiltonian function $H : T^*Q \times \mathbb{R} \to \mathbb{R}$ (see the diagram below).

$$
\begin{tikzcd}
T^*Q \times \mathbb{R} \arrow[d, "\rho"] \arrow[dr, "z"] \arrow[dr, bend left, "H"] & \\
T^*Q & \mathbb{R}
\end{tikzcd}
$$

Recall that we have local canonical coordinates $\{q^i, p_i, z\}, i = 1, \ldots, n$ such that the one-form is $\eta_Q = dz - \rho^*\theta_Q$, $\theta_Q$ being the canonical 1-form on $T^*Q$, can be locally expressed as follows

$$\eta_Q = dz - \sum_{i=1}^{n} p_i dq^i. \tag{37}$$

$(T^*Q \times \mathbb{R}, \eta_Q)$ is a contact manifold with Reeb vector field $\mathcal{R} = \frac{\partial}{\partial z}$.

Consider the Hamiltonian vector field $X_H$ for a given Hamiltonian function, say

$$X_H = \sharp_\Lambda(dH) + H\mathcal{R}. \tag{38}$$

In coordinates, it reads

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \sum_{i=1}^{n}\left(p_i\frac{\partial H}{\partial z} + \frac{\partial H}{\partial q^i}\right)\frac{\partial}{\partial p_i} + \sum_{i=1}^{n}\left(p_i\frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z} \tag{39}$$

We also have

$$\bar{\flat}(X_H) = dH - (\mathcal{R}(H) + H)\eta,$$

where $\flat$ is the isomorphism previously defined. We also have that

$$\eta(X_H) = -H. \tag{40}$$

Recall that $(T^*Q \times \mathbb{R}, \Lambda, -\mathcal{R})$ is a Jacobi manifold with $\Lambda$ given in the usual way.

The contact structure provides us with the *contact Hamilton equations*.

$$\begin{cases} \dot{q}^i = \dfrac{\partial H}{\partial p_i}, \\ \dot{p}_i = -\dfrac{\partial H}{\partial q^i} - p_i \dfrac{\partial H}{\partial z}, \\ \dot{z} = p_i \dfrac{\partial H}{\partial p_i} - H. \end{cases} \tag{41}$$

for all $i = 1, \ldots, n$.
Consider $\gamma$ a section of $\pi : T^*Q \times \mathbb{R} \to Q \times \mathbb{R}$, i.e., $\pi \circ \gamma = \mathrm{id}_{Q \times \mathbb{R}}$. We can use $\gamma$ to project $X_H$ on $Q \times \mathbb{R}$ just defining a vector field $X_H^\gamma$ on $Q \times \mathbb{R}$ by

$$X_H^\gamma = T_\pi \circ X_H \circ \gamma. \tag{42}$$

The following diagram summarizes the above construction

$$\begin{tikzcd}
T^*Q \times \mathbb{R} \arrow[r, "X_H"] \arrow[d, "\pi"] & T(T^*Q \times \mathbb{R}) \arrow[d, "T\pi"] \\
Q \times \mathbb{R} \arrow[r, "X_H^\gamma"] \arrow[u, bend left, "\gamma"] & T(Q \times \mathbb{R})
\end{tikzcd}$$

Assume that in local coordinates we have

$$(q^i, z) \mapsto \gamma(q^i, z) = (q^i, \gamma_j(q^i, z), z)$$

We can compute $T\gamma(X_H^\gamma)$ and obtain

$$T\gamma(X_H^\gamma) = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} + \left(\frac{\partial H}{\partial p_i}\frac{\partial \gamma_j}{\partial q^i} + \left(\gamma_i \frac{\partial H}{\partial p_i} - H\right)\frac{\partial \gamma_j}{\partial z}\right)\frac{\partial}{\partial p_j} + \left(\gamma_i \frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z} \tag{43}$$

Therefore, from (39) and (43), we have that

$$X_H \circ \gamma = T\gamma(X_H^\gamma)$$

if and only if

$$\frac{\partial H}{\partial q^j} + \frac{\partial \gamma_j}{\partial q^i} + \frac{\partial H}{\partial p_i}\gamma_j\frac{\partial H}{\partial z} + \gamma_i\frac{\partial \gamma_j}{\partial z}\frac{\partial H}{\partial p_i} - H\frac{\partial \gamma_j}{\partial z} = 0. \tag{44}$$

Assume now that

1. $\gamma(Q \times \mathbb{R})$ is a coisotropic submanifold of $(T^*Q \times \mathbb{R}, \eta_Q)$;
2. $\gamma_z(Q)$ is a Lagrangian submanifold of $(T^*Q, \omega_Q)$, for any $z \in \mathbb{R}$, where $\gamma_z(q) = \rho \circ \gamma(q, z)$.

Notice that the above two conditions imply that $\gamma(Q \times \mathbb{R})$ is foliated by Lagrangian leaves $\gamma_z(Q)$, $z \in \mathbb{R}$.

We will discuss the consequences of the above conditions.

The submanifold $\gamma(Q \times \mathbb{R})$ is locally defined by the functions

$$\phi_i = p_i - \gamma_i = 0$$

Therefore, the first condition is equivalent to

$$\frac{\partial \gamma_i}{\partial q^j} - \gamma_j \frac{\partial \gamma_i}{\partial z} - \frac{\partial \gamma_j}{\partial q^i} + \gamma_i \frac{\partial \gamma_j}{\partial z} = 0 \tag{45}$$

If, in addition, $\gamma_z(Q)$ is Lagrangian submanifold for any fixed $z \in \mathbb{R}$, then we obtain

$$\frac{\partial \gamma_i}{\partial q^j} - \frac{\partial \gamma_j}{\partial q^i} = 0 \tag{46}$$

and, using again (45), we get

$$\gamma_j \frac{\partial \gamma_i}{\partial z} - \gamma_i \frac{\partial \gamma_j}{\partial z} = 0 \tag{47}$$

Under the above conditions (using (46) and (47)), (44) becomes

$$\frac{\partial H}{\partial q^j} + \frac{\partial H}{\partial p_i}\frac{\partial \gamma_i}{\partial q^j} + \gamma_j \left( \frac{\partial H}{\partial z} + \frac{\partial H}{\partial p_i}\frac{\partial \gamma_i}{\partial z} \right) - H\frac{\partial \gamma_j}{\partial z} = 0. \qquad (48)$$

We can write down eq (48) in a more friendly way. First of all, consider the following functions and 1-forms defined on $Q \times \mathbb{R}$:

1. $$\gamma_o = \frac{\partial H}{\partial z} + \frac{\partial H}{\partial p_i}\frac{\partial \gamma_i}{\partial z}$$

2. $$d(H \circ \gamma_z) = \left( \frac{\partial H}{\partial q^j} + \frac{\partial H}{\partial p_i}\frac{\partial \gamma_i}{\partial q^j} \right) dq^j$$

3. $$i_{\frac{\partial}{\partial z}}(d(\gamma^* \theta_Q)) = \frac{\partial \gamma_j}{\partial z} dq^j$$

Therefore, eq (48) is equivalent to

$$d(H \circ \gamma_z) + \gamma_o(\gamma^* \theta_Q) - (H \circ \gamma)(i_{\frac{\partial}{\partial z}}(d(\gamma^* \theta_Q))) = 0. \qquad (49)$$

**Theorem**

*Assume that a section $\gamma$ of the projection $T^*Q \times \mathbb{R} \longrightarrow Q \times \mathbb{R}$ is such that $\gamma(Q \times \mathbb{R})$ is a coisotropic submanifold of $(T^*Q \times \mathbb{R}, \eta_Q)$, and $\gamma_z(Q)$ is a Lagrangian submanifold of $(T^*Q, \omega_Q)$, for any $z \in \mathbb{R}$. Then, the vector fields $X_H$ and $X_H^\gamma$ are $\gamma$-related if and only if (48) holds (equivalently, (49) holds).*

Equations (48) and (49) are indistinctly referred as a *Hamilton–Jacobi equation with respect to a contact structure*. A section $\gamma$ fullfilling the assumptions of the theorem and the Hamilton-Jacobi equation will be called a *solution* of the Hamilton–Jacobi problem for $H$.

Notice that if $\gamma$ is a solution of the Hamilton–Jacobi problem for $H$, then $X_H$ is tangent to the coisotropic submanifold $\gamma(Q \times \mathbb{R})$, but not necesarily to the Lagrangian submanifolds $\gamma_z(Q)$, $z \in \mathbb{R}$. This occurs when

$$X_H(z - z_0) = 0$$

for any $z_0$, that is, if and only if

$$H \circ \gamma_{z_0} = \gamma_i \frac{\partial H}{\partial p_i}$$

We call $\gamma$ an *strong solution* of the Hamilton–Jacobi problem.

A characterization of conditions on the submanifolds $\gamma(TQ\times\mathbb{R}), \gamma_z(TQ)$ can be given as follows. Let $\sigma : Q\times\mathbb{R}\to\Lambda^k(T^*Q)$ be a $z$-dependent $k$-form on $Q$. Let $d_Q\sigma$ be the exterior derivative at fixed $z$, that is

$$d_Q\sigma(q^i,z) = d\sigma_z(q^i), \tag{50}$$

where $\sigma_z = \sigma(\cdot, z)$. In local coordinates, we have

$$\begin{aligned} d_Q f &= \frac{\partial f}{\partial q^i} dq^i, \\ d_Q(\alpha_i dq^i) &= \frac{\partial \alpha_j}{\partial q^i} dq^i \wedge dq^j, \end{aligned} \tag{51}$$

where $f : Q\times\mathbb{R}\to\mathbb{R}$ is a function and $\alpha = \alpha_i dq^i : Q\times\mathbb{R}\to\Lambda^1(T^*Q)$ is a $z$-dependent 1-form.

**Theorem**

*Let $\gamma$ be a section of $T^*Q\times\mathbb{R}$ over $Q\times\mathbb{R}$. Then $\gamma(Q\times\mathbb{R})$ is a coisotropic submanifold and $\gamma_{z_0}(TQ)$ are Lagrangian submanifolds for all $z_0$ if and only if $d_Q\gamma = 0$ and $\mathcal{L}_{\frac{\partial}{\partial z}}\gamma = \sigma\gamma$ for some function $\sigma : Q\times\mathbb{R}\to\mathbb{R}$. That is, there exists locally a function $f : Q\times\mathbb{R}\to\mathbb{R}$ such that $d_Q f = \gamma$ and $d_Q\frac{\partial f}{\partial z} = \sigma d_Q f$.*

**Proof**
Fix $z_0 \in \mathbb{R}$, then, $\gamma_{z_0}(Q)$ is Lagrangian if and only if $\gamma_{z_0}$ is closed, hence $d\gamma_{z_0} = 0$, so all $\gamma_{z_0}(Q)$ are Lagrangian if and only if $d_Q\gamma = 0$. By the Poincar Lemma, locally $\gamma = d_Q f$,
Now also assume that $\gamma(Q \times \mathbb{R})$ is coisotropic. Then, equation (47) can be written as

$$\gamma \wedge \partial/\partial z\gamma = 0, \tag{52}$$

or, equivalently, that $\gamma$ and $\partial/\partial z\gamma$ are proportional.
Locally, we obtain that $d_Q \frac{\partial f}{\partial z} = \sigma d_Q f$.

## Complete solutions

Next, we shall discuss the notion of complete solutions of the Hamilton–Jacobi problem for a Hamiltonian $H$.

### Definition

A *complete solution* of the Hamilton–Jacobi equation for a Hamiltonian $H$ is a diffeomorphism $\Phi : Q \times \mathbb{R} \times \mathbb{R}^n \to T^*Q \times \mathbb{R}$ such that for any set of parameters $\lambda \in \mathbb{R}^n, \lambda = (\lambda_1, \ldots, \lambda_n)$, the mapping

$$\begin{array}{rcl} \Phi_\lambda : Q \times \mathbb{R} & \to & T^*Q \times \mathbb{R} \\ (q^i, z) & \mapsto & \Phi_\lambda(q^i, z) = \Phi(q^i, z, \lambda) \end{array} \tag{53}$$

is a solution of the Hamilton–Jacobi equation. If, in addition, any $\Phi_\lambda$ is strong, then the complete solution is called an strong complete solution.

We define functions $f_i$ such that for a point $p \in T^*Q \times \mathbb{R}$, it is satisfied

$$f_i(p) = \pi_i \circ \alpha \circ \Phi^{-1}(p). \tag{54}$$

and $\alpha : Q \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is the canonical projection.

The first immediate result is that

$$\mathrm{Im}\ \Phi_\lambda = \cap_{i=1}^{n} f_i^{-1}(\lambda_i)$$

where $\lambda = (\lambda_1, \cdots, \lambda_n)$. In other words,

$$\mathrm{Im}\ \Phi_\lambda = \{x \in T^*Q \times \mathbb{R} \mid f_i(x) = \lambda_i, i = 1, \cdots, n\}$$

Therefore, since $X_H$ is tangent to any of the submanifolds $\mathrm{Im}\ \Phi_\lambda$, we deduce that

$$X_H(f_i) = 0$$

So, these functions are conserved quantities.
Moreover, we can compute

$$\{f_i, f_j\} = \Lambda(df_i, df_j) - f_i\mathcal{R}(f_j) + f_j\mathcal{R}(f_i)$$

But

$$\Lambda(df_i, df_j) = \sharp_\Lambda(df_i)(f_j) = 0$$

since $(T\mathrm{Im}\Phi_\lambda)^\perp = \sharp_\Lambda((T\mathrm{Im}\Phi_\lambda)^o) \subset T\mathrm{Im}\Phi_\lambda$, so

$$\{f_i, f_j\} = -f_i\mathcal{R}(f_j) + f_j\mathcal{R}(f_i) \tag{55}$$

**Theorem**

*There exist no linearly independent commuting set of first-integrals in involution* (68) *for a complete strong solution of the Hamilton–Jacobi equation.*

**Proof:** If all the particular solutions are strong, then the Reeb vector field $\mathcal{R}$ will be transverse to the coisotropic submanifold $\Phi_\lambda(Q \times \mathbb{R})$. Indeed, if $\mathcal{R}$ is tangent to that submanifold, we would have

$$\mathcal{R}(p_i - (\Phi_\lambda)_i) = -\frac{\partial(\Phi_\lambda)_i}{\partial z}$$

where $\Phi_\lambda(q^i, z) = (q^i, (\Phi_\lambda)_i, z)$. So, $\Phi_\lambda$ does not depend on $z$, hence it cannot be a diffeomorphism.
Therefore, if the brackets $\{f_i, f_j\}$ vanish, then we woul obtain that the functions $f_i$ cannot be linearly independent. Indeed, we should have

$$f_i\mathcal{R}(f_j) = f_j\mathcal{R}(f_i)$$

for all $i, j$. But this would imply that $f_i$ and $f_j$ are linearly dependent in the case $\lambda = (0, \ldots, 0)$.

## An alternative approach

Instead of considering sections of $\pi : T^*Q \times \mathbb{R} \longrightarrow Q \times \mathbb{R}$ as above, we could consider a section of the canonical projection $\pi : T^*Q \times \mathbb{R} \longrightarrow Q$, say $\gamma : Q \to T^*Q \times \mathbb{R}$.
In local coordinates, we have

$$(q^i) \mapsto \gamma(q^i) = (q^i, \gamma_j(q^i), \gamma_z(q^i))$$

We want $\gamma$ to fulfill

$$X_H \circ \gamma = T\gamma \circ X_H^\gamma, \tag{56}$$

where $X_H^\gamma = T\pi \circ X_H \circ \gamma$. Using the local expression of $X_H$ we have $X_H^\gamma = \sum_{i=1}^n \left( \frac{\partial H}{\partial p_i} \circ \gamma \right) \frac{\partial}{\partial q^i}$, and equation (56) holds if and only if:

$$-\left(\gamma_i \frac{\partial H}{\partial z} + \frac{\partial H}{\partial q_i}\right) = \sum_{j=1}^{n} \frac{\partial H}{\partial p_j} \frac{\partial \gamma_i}{\partial q^j}, \quad i = 1, \ldots, n, \tag{57}$$

$$\sum_{i=1}^{n} \gamma_i \frac{\partial H}{\partial p_i} - H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \gamma_z q^i. \tag{58}$$

Now, notice that

$$\tilde{\gamma} = \rho \circ \gamma$$

is a 1-form on $Q$. Then, we locally have $\tilde{\gamma} = \gamma_i(q)\, dq^i$.
Next, we assume that $\gamma(Q)$ is a Legendrian submanifold of $(T^*Q \times \mathbb{R}, \eta_Q)$. This implies that $\tilde{\gamma}(Q)$ is a Lagrangian submanifold of $(T^*Q, \omega_Q)$.
But $\gamma(Q)$ is a Legendrian submanifold if and only if it is locally the 1-jet of a function, namely $\gamma = j^1\gamma_z$, where we consider $\gamma_z$ as a function from $Q$ to $\mathbb{R}$. In other words, we have:

$$\gamma_i = \frac{\partial \gamma_z}{\partial q^i} \tag{59}$$

If we assume that the section $\gamma$ fulfills the above condition, we can see that equations (57) become

$$H \circ \gamma = 0. \tag{60}$$

**Definition**

Assume that a section $\gamma$ such that $\gamma(Q)$ is a Legendrian submanifold of $(T^*Q \times \mathbb{R}, \eta_Q)$ and $\tilde{\gamma}(Q)$ is a Lagrangian submanifold of $(T^*Q, \omega_Q)$. Then $\gamma$ is called a solution of the Hamilton-Jacobi problem for the contact Hamiltonian $H$ if equation (60) holds.

## The Hamilton-Jacobi equations for the evolution vector field: A first approach

Assume that $\mathcal{E}_H$ is the evolution vector field defined for a Hamiltonian function $H : T^*Q \times \mathbb{R} \longrightarrow \mathbb{R}$. Then, we have

$$\mathcal{E}_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p_i\frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + p_i\frac{\partial H}{\partial p_i}\frac{\partial}{\partial z} \tag{61}$$

Assume that $\gamma$ is a section of the canonical projection $\pi : T^*Q \times \mathbb{R} \longrightarrow Q \times \mathbb{R}$, say $\gamma : Q \times \mathbb{R} \to T^*Q \times \mathbb{R}$.
In local coordinates we have

$$(q^i, z) \mapsto \gamma(q^i) = (q^i, \gamma_j(q^i), z)$$

Therefore, we can define the projected evolution vector field

$$\mathcal{E}_H^\gamma = T\pi \circ \mathcal{E}_H \circ \gamma.$$

We have that $\mathcal{E}_H \circ \gamma = T\gamma(\mathcal{E}_H^\gamma)$ if and only if

$$\frac{\partial H}{\partial q^j} + \frac{\partial H}{\partial p_i}\frac{\partial \gamma_j}{\partial q^i} + \gamma_i\frac{\partial H}{\partial p_i}\frac{\partial \gamma_j}{\partial z} + \gamma_j\frac{\partial H}{\partial z} = 0 \tag{62}$$

Assume now that

1. $\gamma(Q \times \mathbb{R})$ is a coisotropic submanifold of $(T^*Q \times \mathbb{R}, \eta_Q)$;
2. $\gamma_z(Q)$ is a Legendrian submanifold of $(T^*Q \times \mathbb{R}, \eta_Q)$, for any $z \in \mathbb{R}$, where $\gamma_z(q) = \gamma(q, z)$.

Then, a direct computation shows that (62) becomes

$$d(H \circ \gamma) + \gamma_o \, \gamma^*(\theta_Q) = 0 \, , \tag{63}$$

where

$$\gamma_o = \frac{\partial H}{\partial z} + \frac{\partial H}{\partial p_i} \frac{\partial \gamma_i}{\partial z}$$

**Theorem**

*Assume that a section $\gamma$ of the projection $T^*Q \times \mathbb{R} \longrightarrow Q \times \mathbb{R}$ is such that $\gamma(Q \times \mathbb{R})$ is a coisotropic submanifold of $(T^*Q \times \mathbb{R}, \eta_Q)$, and $\gamma_z(Q)$ is a Legendrian submanifold of $(T^*Q \times \mathbb{R}, \eta_Q)$, for any $z \in \mathbb{R}$. Then, the vector fields $\mathcal{E}_H$ and $\mathcal{E}_H^\gamma$ are $\gamma$-related if and only if (63) holds.*

Equation (63) is referred as a *Hamilton–Jacobi equation for the evolution vector field*. A section $\gamma$ fullfilling the assumptions of the theorem and the Hamilton-Jacobi equation will be called a *solution* of the Hamilton–Jacobi problem for the evolution vector field of $H$.

## An alternative approach

Now $\gamma$ is a section of the canonical projection $\pi : T^*Q \times \mathbb{R} \longrightarrow Q$, say $\gamma : Q \to T^*Q \times \mathbb{R}$. In local coordinates we have

$$(q^i) \mapsto \gamma(q^i) = (q^i, \gamma_j(q^i), \gamma_z(q^i))$$

As in the above sections, we define the projected evolution vector field

$$\mathcal{E}_H^\gamma = T\pi \circ \mathcal{E}_H \circ \gamma.$$

A direct computation shows that $\mathcal{E}_H \circ \gamma = T\gamma(\mathcal{E}_H^\gamma)$ if and only if

$$\frac{\partial H}{\partial q^j} + \frac{\partial H}{\partial p_i}\frac{\partial \gamma_j}{\partial q^i} + \gamma_j \frac{\partial H}{\partial z} = 0 \qquad (64)$$

$$\frac{\partial H}{\partial p_i}\left(\frac{\partial \gamma_z}{\partial q^i} - \gamma_i\right) = 0 \qquad (65)$$

If we assume that $\gamma = j^1 f$, for some function $f : Q \longrightarrow \mathbb{R}$ (or, equivalently, $\gamma(Q)$ is a Legendrian submanifold of $(T^*Q \times \mathbb{R}, \eta_Q)$), then

$$\gamma_i = \frac{\partial \gamma_z}{\partial q^i}$$

and so (64) is fulfilled and (64) becomes

$$d(H \circ \gamma) = 0. \tag{66}$$

Notice that f and $\gamma_z$ define (locally) the same 1-jet.
Therefore, we have the following.

**Theorem**

*Assume that a section $\gamma$ of the projection $T^*Q \times \mathbb{R} \to Q$ is such that $\gamma(Q)$ is a Legendrian submanifold of $(T^*Q \times \mathbb{R}, \eta_Q)$. Then, the vector fields $\mathcal{E}_H$ and $\mathcal{E}_H^\gamma$ are $\gamma$-related if and only if (66) holds.*

Equation (66) is referred as a *Hamilton–Jacobi equation for the evolution vector field*. A section $\gamma$ fullfilling the assumptions of the theorem and the Hamilton-Jacobi equation will be called a *solution* of the Hamilton–Jacobi problem for the evolution vector field of $H$.

## Complete solutions

As in the case of the Hamiltonian vector field, we can consider complete solutions for the evolution vector field.

## Definition

A *complete solution* of the Hamilton–Jacobi equation for the evolution vector field $\mathcal{E}_H$ of a Hamiltonian $H$ on a contact manifold $(M, \eta)$ is a diffeomorphism $\Phi : Q \times \mathbb{R} \times \mathbb{R}^n \to T^*Q \times \mathbb{R}$ such that for any set of parameters $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \in \mathbb{R} \times \mathbb{R}^n$, the mapping

$$\begin{array}{rcl} \Phi_\lambda : Q & \to & T^*Q \times \mathbb{R} \\ (q^i) & \mapsto & \Phi_\lambda(q^i) = \Phi(q^i, \lambda_0, \lambda_1, \ldots, \lambda_n) \end{array} \tag{67}$$

is a solution of the Hamilton–Jacobi equation.

For simplicity, we will use the notation $(\lambda_\alpha\ ,\ \alpha = 0, 1, \ldots, n)$.

As in the previous case, we define functions $f_\alpha$ such that for a point $p \in T^*Q \times \mathbb{R}$, it is satisfied

$$f_\alpha(p) = \pi_\alpha \circ \Phi^{-1}(p). \tag{68}$$

where $\pi_\alpha : Q \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is the canonical projection onto the $\alpha$ factor. A direct computation shows that

$$\mathrm{Im}\ \Phi_\lambda = \cap_{\alpha=0}^{n} f_\alpha^{-1}(\lambda_\alpha)$$

In other words,

$$\mathrm{Im}\ \Phi_\lambda = \{x \in T^*Q \times \mathbb{R} \mid f_\alpha(x) = \lambda_\alpha, \alpha = 0, \cdots, n\}$$

Therefore, since under our hypothesis, $\mathcal{E}_H$ is tangent to any of the submanifolds $\mathrm{Im}\ \Phi_\lambda$, we deduce that

$$\mathcal{E}_H(f_\alpha) = 0$$

So, these functions are conserved quantities for the evolution vector field.

Moreover, we can compute

$$\{f_\alpha, f_\beta\} = \Lambda(df_\alpha, df_\beta) - f_\alpha \mathcal{R}(f_\beta) + f_\beta \mathcal{R}(f_\alpha)$$

But

$$\Lambda(df_\alpha, df_\beta) = \sharp_\Lambda(df_\alpha)(f_\beta) = 0$$

since $(T\mathrm{Im}\Phi_\lambda)^\perp = T\mathrm{Im}\Phi_\lambda$, so

$$\{f_\alpha, f_\beta\} = -f_\alpha \mathcal{R}(f_\beta) + f_\beta \mathcal{R}(f_\alpha) \tag{69}$$

**Theorem**

*There exist no linearly independent commuting set of first-integrals in involution* (68) *for a complete solution of the Hamilton–Jacobi equation for the evolution vector field.*

**Proof:** Since the images of the sections are Legendrian then they are integral submanifolds of ker $\eta_Q$. So, the Reeb vector field $\mathcal{R}$ will be transverse to them, and consequently, there is at least some index $\alpha_0$ such that

$$\mathcal{R}(f_{\alpha_0}) \neq 0$$

Therefore, if all the brackets $\{f_\alpha, f_\beta\}$ vanish, then we woul obtain that the functions $f_\alpha$ cannot be linearly independent.

$\square$

## Examples

**Particle with linear dissipation**
Consider the Hamilltonian $H$

$$H(q,p,z) = \frac{p^2}{2m} + V(q) + \lambda z, \tag{70}$$

where $\lambda \in \mathbb{R}$ is a constant. The extended phase space is $T^*Q \times \mathbb{R} \simeq \mathbb{R}^3$. The Hamiltonian and evolution vector field are given by

$$X_H = \frac{p}{m}\frac{\partial}{\partial q} - \left(\frac{\partial V}{\partial q} + \lambda z\right)\frac{\partial}{\partial p} + \left(\frac{p^2}{2m} - V(q) - \lambda z\right)\frac{\partial}{\partial z}, \tag{71}$$

$$\mathcal{E}_H = \frac{p}{m}\frac{\partial}{\partial q} - \left(\frac{\partial V}{\partial q} + \lambda z\right)\frac{\partial}{\partial p} + \frac{p^2}{m}\frac{\partial}{\partial z}. \tag{72}$$

Assume that $\gamma : Q \to T^*Q \times \mathbb{R}$ is a section of the canonical projection $T^*Q \times \mathbb{R} \to Q$, that is,

$$\gamma(q) = (q, \gamma_p(q), \gamma_z(q)). \tag{73}$$

We assume that $\gamma(Q)$ is a Legendrian submanifold of $T^*Q \times \mathbb{R}$; then,

$$\gamma_p(q) = \frac{\partial \gamma_z}{\partial q}, \tag{74}$$

and $\mathcal{E}_H$ and $\mathcal{E}_H^\gamma$ are $\gamma$-related if and only if

$$H \circ \gamma = k, \tag{75}$$

for a constant $k \in \mathbb{R}$. Then, the HamiltonJacobi equation becomes

$$H(\gamma(q)) = \frac{\gamma_p^2}{2m} + V(q) + \lambda \gamma_z = k, \tag{76}$$

or, equivalently,

$$\frac{\left(\frac{\partial \gamma_z}{\partial q}\right)^2}{2m} + V(q) + \lambda \gamma_z = k, \tag{77}$$

which is a linear ordinary differential equation.
A general solution of the HamiltonJacobi equation (77) is then

$$\gamma_p(q) = \exp(-2m\lambda q) \int (2mk - 2mV(q)) \exp(2m\lambda q) dq. \tag{78}$$

# 6: Applications to thermodynamics

Consider a simple thermodynamic system, for instance, gas in a confined compartment with volume $V$ and pressure $P$ ta temperature $T$.
The state properties of the gas are described by a 2-dimensional submanifold of the thermodynamic phase space, $\mathbb{R}^5$, with coordinates:

- $E$ energy
- $S$ entropy
- $V$, $P$, and $T$

This submanifold is a Legendre submanifold of the contact manifold $(\mathbb{R}^5, \eta)$, where

$$\eta = dE - TdS + PdV$$

Here, $E = E(S, V)$ and the equations of the Legendre submanifold are

$$T = \frac{\partial E}{\partial S}\ ,\ -P = \frac{\partial E}{\partial V}$$

$E, S, V$ are called **extensive** variables
$T, P$ are called **intensive** variables

## Applications to thermodynamics

A.A. Simoes, D. Martín de Diego, M. de León, M. Lainz-Valcázar: Contact geometry for simple thermodynamical systems with friction. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **476** 2241 (2020) doi.org/10.1098/rspa.2020.0244.

A.A. Simoes, D. Martín de Diego, M. de León, M. Lainz-Valcázar: The geometry of some thermodynamic systems. arXiv preprint arXiv:2012.07404 (to appear in an Springer book, 2021).

In this second part, we will introduce a differential geometric framework that incorporates in a very natural way fundamental thermodynamical concepts as the free energy and the rate of entropy production.
Typically, in the previous literature, this description needs to introduce appropriate Poisson and dissipation brackets with combined properties that allows the two laws of thermodynamics to be satisfied.
One of the most successful methods are based on the introduction of *metriplectic structures*:

Allan N. Kaufman. Dissipative Hamiltonian systems: a unifying principle. Phys. Lett. A, 100(8):419–422, 1984.
Philip J. Morrison. A paradigm for joined Hamiltonian and dissipative systems. volume 18, pages 410–419. 1986. Solitons and coherent structures (Santa Barbara, Calif., 1985).

coupling a Poisson and a gradient structure, where the entropy $S$ is now constructed from a Casimir function of the Poisson structure.

Other approaches like in

B. J. Edwards and A. N. Beris. Noncanonical Poisson bracket for nonlinear elasticity with extensions to viscoelasticity. J. Phys. A, 24(11):2461–2480, 1991.
B. J. Edwards and A. N. Beris. Noncanonical Poisson bracket for nonlinear elasticity with extensions to viscoelasticity. J. Phys. A, 24(11):2461–2480, 1991.

use similar techniques, called *single generation formalism* introducing a generalized bracket which is naturally divided into two parts: a non-canonical Poisson bracket and a new dissipation bracket.

The derived structures are capable of reproducing both reversible and irreversible evolutions providing a unifying formalism for many systems ruled by the laws of thermodynamics.

These approaches have proved to be very useful for the description of complex thermodynamical systems and also facilitate their numerical integration.

Recently, Gay-Balmaz and Yoshimura

F. Gay-Balmaz and H. Yoshimura. A Lagrangian variational formulation for nonequilibrium thermodynamics. Part I: Discrete systems. Journal of Geometry and Physics, 111:169–193, January 2017.
F. Gay-Balmaz and H. Yoshimura. From Lagrangian Mechanics to Nonequilibrium Thermodynamics: A Variational Perspective. Entropy, 21(1):8, January 2019.

have introduced a variational principle for the description of thermodynamical systems.

Their formulation extends the Hamilton principle of classical mechanics to include irreversible processes by introducing additional phenomenological and variational constraints.

A more geometrical approach is based on the use of contact geometry; indeed, the thermodynamical phase space is equipped with a contact structure:

- Using the contact structure, it is possible to associate to each function $f$, a Hamiltonian vector field $X_f$ which is the infinitesimal generator of a contact transformation.
- In this framework, the manifold of equilibrium states is represented by a Legendre submanifold $N$ and the Hamiltonian vector field $X_f$ is tangent to $N$ if and only if the function $f$ vanishes on $N$, that is, the Legendre submanifold is contained on the zero level set of the Hamiltonian function.
- The flow of $X_f$ restricted to the Legendrian submanifold is interpreted as thermodynamical processes.

R. Mrugala, J. D. Nulton, J. Ch. Schon, and P. Salamon. Contact structure in thermodynamic theory. Reports on Mathematical Physics, 29(1):109–121, February 1991.
R. Mrugala. Continuous contact transformations in thermodynamics. In Proceedings of the XXV Symposium on Mathematical Physics (Torun, 1992), vol. 33, pages 149–154, 1993.

Another approach to the dynamics of thermodynamical processes is the one used in

R. Balian and P. Valentin. Hamiltonian structure of thermodynamics with gauge. The European Physical Journal B-Condensed Matter and Complex Systems, 21(2):269–282, 2001.
A. Van der Schaft and B. Maschke. Geometry of thermodynamic processes. Entropy, 20(12):925, 2018.

which is based on homogeneous symplectic Hamiltonian systems, and is completely equivalent to the contact Hamiltonian vector field approach.

A recent paper is

Arjan van der Schaft: Liouville geometry of classical thermodynamics. *arXiv:2102.05493*

## Contact geometry and contact dynamics

We will consider some ingredients of contact geometry that we will need in the sequel.

Let $M$ be a differentiable manifold of dimension $2n+1$ and a 1-form $\eta$ on $M$. We say that $\eta$ is a contact 1-form if $\eta \wedge (d\eta)^n \neq 0$ at every point. Then, we call $(M, \eta)$ a contact manifold. A distinguished vector field for a contact manifold is the Reeb vector field $\mathcal{R} \in \mathfrak{X}(M)$ univocally characterized by

$$i_{\mathcal{R}}\eta = 1 \quad \text{and} \quad i_{\mathcal{R}}d\eta = 0 \, .$$

We can define also an isomorphism of $C^\infty(M, \mathbb{R})$ modules by

$$\begin{array}{rccl} \bar{\flat}: & \mathfrak{X}(M) & \longrightarrow & \Omega^1(M) \\ & X & \longmapsto & i_X d\eta + \eta(X)\eta \end{array}$$

Observe that $\bar{\flat}^{-1}(\eta) = \mathcal{R}$.

Using the generalized Darboux theorem, we have canonical coordinates $(q^i, p_i, S)$, $1 \leq i \leq n$ in a neighborhooh of every point $x \in M$, such that the contact 1-form $\eta$ and the Reeb vector field are:

$$\eta = dS - p_i \, dq^i \qquad \text{and} \qquad \mathcal{R} = \frac{\partial}{\partial S} \, .$$

Define the bi-vector $\Lambda$ on $M$ by

$$\Lambda(\alpha, \beta) = -d\eta(\flat^{-1}(\alpha), \flat^{-1}(\beta)), \qquad \alpha, \beta \in \Omega^1(M) \, . \tag{79}$$

In canonical coordinates,

$$\Lambda = \frac{\partial}{\partial p_i} \wedge \left( \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial S} \right) \tag{80}$$

Define the $C^\infty(M, \mathbb{R})$-linear mapping

$$\sharp_\Lambda : \Omega^1(M) \to \mathfrak{X}(M)$$

by $\langle \beta, \sharp_\Lambda(\alpha) \rangle = \Lambda(\alpha, \beta)$ with $\alpha, \beta \in \Omega^1(M)$.

Given a function $f \in C^\infty(M, \mathbb{R})$ we will define the following vector fields

- **Hamiltonian or contact vector field** $X_f$ defined by

$$X_f = \sharp_\Lambda(df) - f\mathcal{R}$$

or in other terms, $X_f$ is the unique vector field such that

$$\bar{\flat}(X_f) = df - (\mathcal{R}(f) + f)\,\eta\,.$$

In canonical coordinates:

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i\frac{\partial f}{\partial S}\right)\frac{\partial}{\partial p_i} + \left(p_i\frac{\partial f}{\partial p_i} - f\right)\frac{\partial}{\partial S}$$

- **The evolution or horizontal vector field**

$$\mathcal{E}_f = \sharp_\Lambda(df) = X_f + f\mathcal{R}$$

or

$$\bar{\flat}(\mathcal{E}_f) = df - \mathcal{R}(f)\,\eta\,.$$

In canonical coordinates:

$$\mathcal{E}_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i\frac{\partial f}{\partial S}\right)\frac{\partial}{\partial p_i} + p_i\frac{\partial f}{\partial p_i}\frac{\partial}{\partial S}$$

## Remarks

1. We will see that the evolution vector field will be useful to describe some simple isolated thermodynamical systems with friction, where the variable $S$ will play the role of the entropy of the system.

2. The interpretation of the variable $S$ as being the entropy of the system excludes the possibility of using cosymplectic geometry to describe thermodynamical systems. Indeed, if the thermodynamical equations were the integral curves of the cosymplectic Hamiltonian vector field, then the entropy production would be constant, which is not the general situation.

## Jacobi and Cartan brackets

The pair $(\Lambda, E = -R)$ is a particular case of Jacobi structure since it satisfies

$$[\Lambda, \Lambda] = 2E \wedge \Lambda \quad \text{and} \quad [\Lambda, E] = 0 \,.$$

From this Jacobi structure we can define the Jacobi bracket as follows:

$$\{f, g\} = \Lambda(df, dg) + fE(g) - gE(f), \quad f, g \in C^\infty(M, \mathbb{R})$$

The mapping $\{\ ,\ \} : C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$ is bilinear, skew-symmetric and satisfies the Jacobi's identity but, in general, it does not satisfy the Leibniz rule; this last property is replaced by a weaker condition:

$$\text{Supp}\ \{f, g\} \subset \text{Supp}\ f \cap \text{Supp}\ g \,.$$

In this sense, this bracket generalizes the well-known Poisson brackets. Indeed, a Poisson manifold is a particular case of Jacobi manifold. In local coordinates

$$\{f, g\} \quad = \quad \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial S}\left(p_i\frac{\partial g}{\partial p_i} - g\right) + \frac{\partial g}{\partial S}\left(p_i\frac{\partial f}{\partial p_i} - f\right)$$

It is also interesting for us to introduce the *Cartan bracket* (that does not obey the Jacobi identity)

$$
\begin{aligned}
[f, g] &= \Lambda(df, dg) \\
&= \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial S}\left(p_i \frac{\partial g}{\partial p_i}\right) + \frac{\partial g}{\partial S}\left(p_i \frac{\partial f}{\partial p_i}\right)
\end{aligned}
$$

## Mechanics

Our main example of contact manifold along this talk will be $T^*Q \times \mathbb{R}$, where $Q$ is $n$-dimensional manifold, with contact structure defined by

$$\eta_Q = pr_2^*(dS) - pr_1^*(\theta_Q) \equiv dS - \theta_Q$$

where $pr_1 : T^*Q \times \mathbb{R} \to T^*Q$ and $pr_2 : T^*Q \times \mathbb{R} \to \mathbb{R}$ are the canonical projections and $\theta_Q$ is the Liouville 1-form on the cotangent bundle defined by

$$\eta_Q(X_{\mu_q}) = \langle \mu_q, T_{\mu_q}\pi_Q X_{\mu_q} \rangle$$

being $X_{\mu_q} \in T_{\mu_q} T^*Q$. Taking bundle coordinates $(q^i, p_i)$ on $T^*Q$ we have that $\eta_Q = dS - p_i dq^i$.
On such a manifold we can define the bi-vector

$$\Lambda_0 = \Lambda + \sharp_\Lambda(dS) \wedge \mathcal{R}$$

which is Poisson, that is $[\Lambda_0, \Lambda_0] = 0$. In coordinates,

$$\Lambda_0 = \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}$$

is like the canonical Poisson bracket on $T^*Q$ but now applied to functions on $T^*Q \times \mathbb{R}$.

Observe that in this case the Cartan bracket can be rewritten in terms of the Poisson bracket induced by $\Lambda_0$ and an extra term that describe the thermodynamical behaviour. That is,

$$[f,g] = \{f,g\}_{\Lambda_0} - \frac{\partial f}{\partial S}\Delta g + \frac{\partial g}{\partial S}\Delta f$$

where $\Delta = -\sharp_\Lambda(dS)$ is the Liouville vector field:

$$\Delta = p_i \frac{\partial}{\partial p_i}$$

We will denote by

$$\{f,g\}_\Delta = \frac{\partial g}{\partial S}\Delta f - \frac{\partial f}{\partial S}\Delta g$$

Then, the Cartan bracket is written as in the single generation formalism as

$$[f,g] = \{f,g\}_{\Lambda_0} + \{f,g\}_\Delta \tag{81}$$

Now, we will discuss some interesting properties of the qualitative behaviour of the evolution vector field $\mathcal{E}_f$.

**Proposition** The Lie derivative of the contact form $\eta$ with respect to the evolution vector field $\mathcal{E}_f$ associated to the Hamiltonian function $f$ satisfies the following relation

$$\mathcal{L}_{\mathcal{E}_f}\eta = -R(f)\eta + df \ .$$

**Proof**:
The proof is a trivial consequence of the properties of the Lie derivative and the properties of the Hamiltonian vector field:

$$\begin{aligned} \mathcal{L}_{\mathcal{E}_f}\eta &= \mathcal{L}_{X_f+fR}\eta = \mathcal{L}_{X_f}\eta + \mathcal{L}_{fR}\eta \\ &= -R(f)\eta + (i_R\eta)df = -R(f)\eta + df \end{aligned}$$

**Theorem** Let $\mathcal{L}_c(f) = f^{-1}(c)$ be a level set of $f : M \to \mathbb{R}$ where $c \in \mathbb{R}$. We assume that $\mathcal{L}_c(f) \neq \emptyset$ and $R(f)(x) \neq 0$ for all $x \in \mathcal{L}_c(f)$. Then

1. The 2-form $\omega_c \in \Omega^2(\mathcal{L}_c(f))$ defined by

$$\omega_c = -di_c^*\eta$$

is an exact symplectic structure. Here $i_c : \mathcal{L}_c f \hookrightarrow M$ denotes the canonical inclusion

2. If $\Delta_c$ is the Liouville vector field, that is,

$$i_{\Delta_c}\omega_c = i_c^*\eta$$

then the restriction of $\mathcal{E}_f$ to $\mathcal{L}_c(f)$ verifies that

$$\mathcal{E}_f\big|_{\mathcal{L}_c(f)} = R(f)\big|_{\mathcal{L}_c(f)}\Delta_c$$

**Proof**:
The form $\omega_c$ is trivially closed. To see that it is a symplectic form, we just need to check that is non degenerate. Let $p \in \mathcal{L}_c(f)$. Notice that, at that point, $\omega_c = -d\eta|_{T_p\mathcal{L}_c(f)}$. By the condition $R(f) \neq 0$, we have that $R_p$ (and, hence $\ker \eta = \text{span}\, \langle R \rangle$) is transverse to $T_p\mathcal{L}_c(f)$. But since $\eta_p \wedge d\eta_p^n \neq 0$,then $d\eta|_V$ is non-degenerate for every subspace $V$ transverse to $\ker \eta$. Therefore, $\omega_c$ is also non-degenerated.
For the second part, we first remark that $\mathcal{E}_f(f) = 0$, hence $(i_c)_* \mathcal{E}_f = \mathcal{E}_f|_{\mathcal{L}_c(f)}$ is a well-defined vector field. By the above Proposition and Cartan's identity

$$i_{\mathcal{E}_f} d\eta = -R(f)\eta + df.$$

Pulling back by $i_c$, we get

$$i_{(i_c)_* \mathcal{E}_f} i_c^* d\eta = -(R(f) \circ i_c) i_c^* \eta + d i_c^* f = -(R(f) \circ i_c) i_c^* \eta,$$

dividing by $-(R(f) \circ i_c)$,

$$-i_{(i_c)_* \mathcal{E}_f / R(f)} i_c^* d\eta = i_{(i_c)_* \mathcal{E}_f / R(f)} \omega_c = i_c^* \eta.$$

Thus, $(i_c)_* \left(\mathcal{E}_f / R(f)\right) = \Delta_c$, as we wanted to show.

## Remarks

1. Observe that since

$$\mathcal{E}_f\big|_{\mathcal{L}_c(f)} = R(f)\big|_{\mathcal{L}_c(f)}\Delta_c$$

then, the dynamics on each energy level is like a Liouville dynamics after a time reparametrization

$$dt = \frac{1}{R(f)}d\tau \ .$$

2. It is interesting to note that $T^*Q \times \mathbb{R}$ is also the phase space for time-dependent dynamics. In this case, the appropriate formalism is the cosymplectic formalism where the canonical cosymplectic structure is given by $(dt, \omega_Q)$

A. Bravetti, M. de León, J. C. Marrero, E. Padrón: Invariant measures for contact Hamiltonian systems: symplectic sandwiches with contact bread. *J. Phys. A* 53 (2020), no. 45, 455205, 24 pp.

- We prove that, under some natural conditions, Hamiltonian systems on a contact manifold C can be split into a Reeb dynamics on an open subset of C and a Liouville dynamics on a submanifold of C of codimension 1.
- For the Reeb dynamics we find an invariant measure.
- Moreover, we show that, under certain completeness conditions, the existence of an invariant measure for the Liouville dynamics can be characterized using the notion of a symplectic sandwich with contact bread.

## Simple thermodynamical systems

We will use the evolution vector field to describe simple thermodynamical systems, that is, thermodynamical systems whose configuration space is composed by just one scalar thermal variable (in our case the entropy) and a finite set of mechanical variables (position and momenta). We will assume that the system is isolated, that is, there is not any transfer of work, matter or heat.

The isolated simple thermodynamical systems are described by a Lagrangian function:

$$\begin{array}{rccc} L: & TQ \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ & (v_q, S) & \longmapsto & L(v_q, S) \end{array}$$

where $Q$ is the configuration manifold describing the mechanical part of the thermodynamical system, $TQ$ is the tangent bundle with canonical projection $\tau_Q : TQ \to Q$ given by $\tau_Q(v_q) = q$. The entropy of the system is described by the real variable $S \in \mathbb{R}$. If we consider coordinates $(q^i)$ on $Q$ and induced coordinates $(q^i, \dot{q}^i)$ on $TQ$, then $\tau_Q(q^i, \dot{q}^i) = (q^i)$.

We will see that the Lagrangian function itself will produce a friction force satisfying naturally the two laws of thermodynamics.

We will assume that the Lagrangian system is regular, that is, the matrix

$$(W_{ij}) = \left( \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right)$$

is regular or, equivalently, the mapping $\mathbb{F}L : TQ \times \mathbb{R} \to T^*Q \times \mathbb{R}$ is a local diffeomorphism, where:

$$\mathbb{F}L(q^i, \dot{q}^i, S) = \left( q^i, \frac{\partial L}{\partial \dot{q}^i}, S \right)$$

is the Legendre transform. For simplicity, we will assume that the Legendre transform is a global diffeomorphism, since if it was only a local diffeomorphism we could proceed analogously by restricting to a neighbourhood. Then, we may define a Hamiltonian function $H : T^*Q \times \mathbb{R} \to \mathbb{R}$ given by

$$H(q^i, p_i, S) = p_i \dot{q}^i - L(q^i, \dot{q}^i, S)$$

where now the coordinates $\dot{q}^i$ are implicitly defined by the relations $p_j = \frac{\partial L}{\partial \dot{q}^j}(q^i, \dot{q}^i, S)$.

The equations of motion defined by the evolution vector field $\mathcal{E}_H$ are

$$
\begin{aligned}
\frac{dq^i}{dt} &= \frac{\partial H}{\partial p_i}\,, \\
\frac{dp_i}{dt} &= -\frac{\partial H}{\partial p_i} - p_i \frac{\partial H}{\partial S} \\
\frac{dS}{dt} &= p_i \frac{\partial H}{\partial p_i}\,.
\end{aligned}
$$

The vector field $\mathcal{E}_H$ satisfies the following two properties that correspond to the first and second laws of thermodynamics: conservation of the energy of an isolated system and irreversibility of the processes, that is, non-decreasing entropy production.

**Proposition** The integral curves of $\mathcal{E}_H$ satisfies the following properties:

1. $\mathcal{E}_H(H) = 0$, that is, $\frac{dH}{dt} = 0$;
2. $\mathcal{E}_H(S) = \Delta(H)$, that is, $\frac{dS}{dt} = \Delta H$.

**Proof** Both are a direct consequence of the definition of the evolution vector field $\mathcal{E}_H = \sharp_\Lambda(dH)$.

Assume that the Hamiltonian $H$ is given by

$$H(q^i, p_i, S) = \frac{1}{2} g^{ij} p_i p_j + V(q, S) \tag{82}$$

where $(g^{ij})$ is positive semi-definite (for instance, it is associated to a Riemannian metric on $Q$). Then, the vector field $\mathcal{E}_H$ describes an isolated simple thermodynamical system with friction satisfying the first and second laws of thermodynamics:

**Proposition** The integral curves of $\mathcal{E}_H$ satisfies the following properties:

1. **First law of Thermodynamics**:

$$\frac{dH}{dt} = 0 \quad \text{(preservation of the total energy);}$$

2. **Second law of Thermodynamics**:

$$\frac{dS}{dt} = \Delta H \geq 0 \quad \text{(total entropy of an isolated system never decreases).}$$

**Proof** It is a direct consequence of the above Proposition and $\Delta H = p_i g^{ij} p_j \geq 0$.

If we express the dynamics in terms of the brackets defined in (81) we have that

$$\dot{f} = \{f, H\}_{T^*Q} + \{f, H\}_{\Delta}. \tag{83}$$

Obviously,

$\{H, H\}_{T^*Q} = \{H, H\}_{\Delta} = 0$ (first law)

and

$\{S, H\}_{T^*Q} = 0$ and $\{S, H\}_{\Delta} = \Delta H \geq 0$ (second law).

Observe that in Equation (83) both brackets are using the function $H$ as "generator". This is the reason that typically this formalism is known as *single generator formalism*.

## Linearly damped systems

Consider a linearly damped system described by coordinates $(q, p, S)$, where $q$ represents the position, $p$ the momentum of the particle and $S$ is the entropy of the surrounding thermal bath. We assume that the system is subjected to a viscous friction force, proportional to the minus velocity of the particle. The system is described by the Hamiltonian

$$H(q, p, S) = \frac{p^2}{2m} + V(q) + \gamma S, \quad \gamma > 0.$$

Therefore, the equations of motion for $\mathcal{E}_H = \sharp_\Lambda(dH)$ are:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \\ \dot{S} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -p \\ 0 & p & 0 \end{pmatrix} \begin{pmatrix} V'(q) \\ p/m \\ \gamma \end{pmatrix}$$

or

$$\begin{aligned} \dot{q} &= \frac{p}{m} \\ \dot{p} &= -V'(q) - \gamma p \\ \dot{S} &= \frac{p^2}{m} \end{aligned}$$

In the Lagrangian side we obtain the system given by

$$\begin{aligned} m\ddot{q} &= -V'(q) - \gamma m\dot{q} \\ \dot{S} &= m\dot{q}^2. \end{aligned}$$

Observe that in this system the friction force is given by the map $F_{fr} : TQ \to T^*Q$ given by

$$F_{fr}(q, \dot{q}) = \gamma \dot{q}^i dq^i.$$

Therefore, the equation of entropy production can be rewritten in terms of the friction force as follows

$$T\dot{S} = -\langle F_{fr}(q, \dot{q}), \dot{q} \rangle$$

where $T = \frac{\partial H}{\partial S} = -\frac{\partial L}{\partial S} = \gamma > 0$ represents the temperature of the thermal bath. These equations coincide with the set of equations proposed by Gay-Balmaz and Yoshimura for this particular choice of Lagrangian $L$ and friction force $F_{fr}$. Observe that, in this particular example where the temperature satisfies $T = \gamma$, the equations are only defined for values $\gamma > 0$ and thus we are only modelling thermodynamical systems with non-zero temperature.

Observe that the two brackets are:

$$\begin{aligned}
\{f,g\}_{\Lambda_0} &= \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q} \\
\{f,g\}_{\Delta} &= p\frac{\partial g}{\partial S}\frac{\partial f}{\partial p} - p\frac{\partial f}{\partial S}\frac{\partial g}{\partial p}
\end{aligned}$$

In particular

$$\begin{aligned}
\{H,g\}_{\Lambda_0} &= \frac{p}{m}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}V'(q) \\
\{H,g\}_{\Delta} &= \frac{p^2}{m}\frac{\partial g}{\partial S} - \gamma p\frac{\partial g}{\partial p}
\end{aligned}$$

and

$$\mathcal{E}_H(g) = \dot{g} = \{H,g\}_{\Lambda_0} + \{H,g\}_{\Delta}$$

Therefore it is clear that $\{H,H\}_{\Lambda_0} = 0$ and $\{H,H\}_{\Delta} = 0$ (by skew-symmetry) and $\{H,S\}_{\Lambda_0} = 0$ and $\{H,S\}_{\Delta} = \frac{p^2}{m} \geq 0$.

## Composed thermodynamical systems without friction

We will present a model for systems composed of at least two subsystems exchanging heat with each other.
Consider two thermodynamic systems indexed by 1 and 2 which may interact through a conducting wall. On each system we have defined the corresponding Hamiltonian:

$$H_i : T^*Q_i \times \mathbb{R} \longrightarrow \mathbb{R} , \qquad 1 \leq i \leq 2$$

where $(q_i, p_i, S_i)$ are coordinates on $T^*Q_i \times \mathbb{R}$, $i = 1, 2$, and $S_i$ are the entropies of each subsystem.
Consider the total energy $H : T^*(Q_1 \times Q_2) \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$H(q_1, p_1, q_2, p_2, S_1, S_2) = H_1(q_1, p_1, S_1) + H_1(q_1, p_1, S_1)$$

and the symplectic 2-form on $T^*(Q_1 \times Q_2) \times \mathbb{R}^2$

$$\Omega_{Q_1 \times Q_2 \times \mathbb{R}} = \omega_{Q_1} + \omega_{Q_2} + dS_1 \wedge dS_2$$

Denote by $\Lambda_{Q_1 \times Q_2 \times \mathbb{R}}$ the corresponding Poisson tensor.

Assume that both subsystems exchange heat according to Fourier Law:

$$h = k(T_2 - T_1)$$

where $T_i$ is the absolute temperature of subsystem $i$, $1 \leq i \leq 2$ and $k$ is the coefficient of thermal conductivity. We have that $T_i = \frac{\partial H}{\partial S_i} > 0$.
Consider the function $K : T^*(Q_1 \times Q_2) \times \mathbb{R}^2 \to \mathbb{R}$

$$K = k\left(\frac{1}{\frac{\partial H}{\partial S_1}} - \frac{1}{\frac{\partial H}{\partial S_2}}\right) = k\left(\frac{1}{T_1} - \frac{1}{T_2}\right)$$

which will be called *Fourier factor*. Define the Hamiltonian vector field of $K$ by

$$X_K = \Lambda_{Q_1 \times Q_2 \times \mathbb{R}}(\cdot, dK) \quad \text{or alternatively} \quad i_{X_K}\Omega_{Q_1 \times Q_2 \times \mathbb{R}} = dK\ .$$

Define the two-tensor (with Fourier factor $K$) denoted by $\Lambda_K$, which is associated in the canonical way with the 2-form

$$\Omega_K = \omega_{Q_1} + \omega_{Q_2} + K dS_1 \wedge dS_2$$

Observe that now $\Omega_K$ is no longer a symplectic form and so $\Lambda_K$ is a skew-symmetric Poisson structure. In other words, $\Lambda_K$ is a skew-symmetric tensor.

The matrix representation of $\Lambda_K$ is:

$$\begin{pmatrix} 0 & I_{n_1\times n_1} & 0 & 0 & 0 & 0 \\ -I_{n_1\times n_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_2\times n_2} & 0 & 0 \\ 0 & 0 & -I_{n_2\times n_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k\left(\frac{1}{T_1}-\frac{1}{T_2}\right) \\ 0 & 0 & 0 & 0 & k\left(\frac{1}{T_2}-\frac{1}{T_1}\right) & 0 \end{pmatrix}$$

where $\dim Q_i = n_i,\ i = 1, 2.$

The corresponding evolution vector field $\mathcal{E}_{H,K}$:

$$\mathcal{E}_{H,K} = \sharp_{\Lambda_K}(dH) \tag{84}$$

The integral curves of $\mathcal{E}_{H,K}$ are:

$$\begin{aligned} \dot{q}_1 &= \frac{\partial H}{\partial p_1} & \dot{q}_2 &= \frac{\partial H}{\partial p_2} \\ \dot{p}_1 &= -\frac{\partial H}{\partial q_1} & \dot{p}_2 &= -\frac{\partial H}{\partial q_2} \\ \dot{S}_1 &= k(\frac{T_2}{T_1} - 1) & \dot{S}_2 &= k(\frac{T_1}{T_2} - 1). \end{aligned} \tag{85}$$

Observe that the total entropy $S = S_1 + S_2$ satisfies

$$\begin{aligned} \dot{S} &= \mathcal{E}_{H,K}(S_1 + S_2) \\ &= k(\frac{T_2}{T_1} - 1) + k(\frac{T_1}{T_2} - 1) \\ &= k\frac{(T_2 - T_1)^2}{T_1 T_2} \geq 0 \end{aligned}$$

Moreover, in absence of external forces, the total energy $H$ is conserved since by skew-symmetry of $\Lambda_K$ we have that

$$\mathcal{E}_{H,K}(H) = 0 \ .$$

Thus, we have shown that:

**Proposition**

*Given a Hamiltonian function $H$, the evolution vector field $\mathcal{E}_{H,K}$ in the skew-symmetric manifold $(T^*(Q \times Q \times \mathbb{R}), \Lambda_K)$, which encodes the dynamics of composed thermodynamical systems exchanging heat with each other, satisfies the first and second laws of Thermodynamics.*

## Example

The toy model for this case is the two free thermo-particles example, composed by two particles of equal mass such that the system is modelled by the Hamiltonian function

$$H(q_1, p_1, S_1, q_2, p_2, S_2) = \frac{1}{2m}(p_1^2 + p_2^2) + T_1 S_1 + T_2 S_2,$$

where $m$ is the mass of both particles and $T_1$, $T_2$ are the temperatures of each particle.

A physical example is the two thermo-spring system, where the system is modelled by the Hamiltonian function

$$H(q_1, p_1, S_1, q_2, p_2, S_2) = \frac{1}{2m}(p_1^2 + p_2^2) + V(q_1, q_2) + T_1 S_1 + T_2 S_2,$$

where $V$ is the potential energy associated to the interplay between both springs, $m$ is the mass of both springs and $T_1$, $T_2$ are the temperatures of each spring.

## Other topics in contact Hamiltonian dynamics

In the next slides I will indicate some of the research topics we have been working on in recent years. I will indicate only the topic, a very brief description and the references where you can find more details.

## Infinitesimal symmetries and dissipate quantities

M. de León, M. Lainz-Valcázar: Infinitesimal symmetries in contact Hamiltonian systems. *J. Geom. Phys*. **153** (2020), 103651, 13 pp
M. de León, M. Lainz-Valcázar, A. López-Gordon: Symmetries, constants of the motion, and reduction of mechanical systems with external forces. *J. Math. Phys.* **62** (2021), no. 4, 042901, 16 pp.

1. In the first paper, we extend the well-known Noether theorem for Lagrangian systems to contact Lagrangian systems. Moreover, we introduce a classification of infinitesimal symmetries and obtain the corresponding dissipated quantities. We notice that in contact dynamics, the existence of infinitesimal symmetries does not produce conserved quantities, but functions that dissipate at the same rate than the energy; so, the corresponding quotients are true conserved quantities. The case of contact reduction by a Lie group of symmetries is also discussed.
2. In the second paper we consider symmetries for non-conservative systems, including Rayleigh dissipation.

## Contact nonholonomic mechanical systems

M. de León, V.M. Jiménez, M. Lainz-Valcázar: Contact Hamiltonian and Lagrangian systems with nonholonomic constraints. *Journal of Geometric Mechanics* (2021) doi: 10.3934/jgm.2021001.

- In this article we develop a theory of contact systems with nonholonomic constraints.
- We obtain the dynamics from Herglotzs variational principle, by restricting the variations so that they satisfy the nonholonomic constraints.
- We prove that the nonholonomic dynamics can be obtained as a projection of the unconstrained Hamiltonian vector field.
- Finally, we construct the nonholonomic bracket, which is an almost Jacobi bracket on the space of observables and provides the nonholonomic dynamics.

## Singular contact Lagrangian systems

M. de León, M. Lainz-Valcázar: Singular Lagrangians and precontact Hamiltonian systems. *Int. J. Geom. Methods Mod. Phys.* **16** (2019), no. 10, 1950158, 39 pp.

- In the first paper we discuss singular Lagrangian systems on the framework of contact geometry. These systems exhibit a dissipative behavior in contrast with the symplectic scenario.
- We develop a constraint algorithm similar to the presymplectic onestudied by Gotay and Nester (the geometrization of the well-known Dirac-Bergman algorithm).
- We also construct the Hamiltonian counterpart and prove the equivalence with the Lagrangian side.
- A Dirac-Jacobi bracket is constructed similar to the Dirac bracket.

## Contact discrete dynamics

A.A. Simoes, D. Martín de Diego, M. de León, M. Lainz-Valcázar: On the geometry of discrete contact mechanics. *Journal of Nonlinear Sciencer* 31 (2021), no. 3, Paper No. 53.

- We introduce a discrete Herglotz Principle and the corresponding discrete Herglotz Equations for a discrete Lagrangian in the contact setting. This allows us to develop convenient numerical integrators for contact Lagrangian systems that are conformally contact by construction.
- The existence of an exact Lagrangian function is also discussed.

## Uniform formalism

M. de León, J. Gaset, M. Lainz-Valcázar, X. Rivas, N. Román-Roy: Unified Lagrangian-Hamiltonian formalism for contact systems. *Fortschr. Phys*. **68** (2020), no. 8, 2000045, 12 pp.

- We develop a unified geometric framework for describing both the Lagrangian and Hamiltonian formalisms of contact autonomous mechanical systems, which is based on the approach of the pionnering work of R. Skinner and R. Rusk.
- This framework permits to skip the second order differential equation problem, which is obtained as a part of the constraint algorithm (for singular or regular Lagrangians), and is specially useful to describe singular Lagrangian systems.
- Some examples are also discussed to illustrate the method.

## Contact Optimal Control Theory

M. de León, M. Lainz-Valcázar, M.C. Muñoz-Lecanda: Optimal control, contact dynamics and Herglotz variational problem *arXiv preprint arXiv:2006.14326* (2020)

- We combine two main topics in mechanics and optimal control theory: contact Hamiltonian systems and Pontryagin Maximum Principle.
- As an important result, a contact Pontryagin Maximum Principle that permits to deal with optimal control problems with dissipation is developed.
- Also, the Herglotz optimal control problem is stated, in such a way that generalizes simultaneously the Herglotz variational principle and an optimal control problem.
- Some applications to the study of a thermodynamic system are provided.

## Existence of invariant measures

A. Bravetti, M. de León, J.C. Marrero, E. Padrón: Invariant measures for contact Hamiltonian systems: symplectic sandwiches with contact bread *Journal of Physics A: Mathematical and Theoretical* **53** (45), (2020) 455205.

- An important topic in dynamical systems is the existence of invariant measures. We prove that, under some natural conditions, Hamiltonian systems on a contact manifold $C$ can be split into a Reeb dynamics on an open subset of $C$ and a Liouville dynamics on a submanifold of $C$ of codimension 1.
- Thus, an invariant measure is obianed for the Reeb dynamics, and moreover, a under certain completeness conditions, the existence of an invariant measure for the Liouville dynamics can be characterized using the notion of a symplectic sandwich with contact bread developed in this paper.

## Contact higher order mechanics

M. de León, J. Gaset, M Lainz-Valcázar, M.C. Muñoz-Lecanda, N. Román-Roy: Higher-order contact mechanics. *Annals of Physics* **425** 168396 (2021)

- We present a complete theory of higher-order autonomous contact mechanics, which allows us to describe higher-order dynamical systems with dissipation.
- The essential tools for the theory are the extended higher-order tangent bundles, $T^k Q \times \mathbb{R}$, and its canonical geometric structures. This allow us to state the Lagrangian and Hamiltonian formalisms for these kinds of systems, as well as their variational formulation.
- In that paper, a unified description that encompasses the Lagrangian and Hamiltonian equations as well as their relationship through the Legendre map; all of them are obtained from the contact dynamical equations and the constraint algorithm that is implemented because, in this formalism, the dynamical systems are always singular.
- The theory is applied to some interesting examples.

## Inverse problem for contact Lagrangian mechanics

M. de León, J. Gaset, M. Lainz-Valcárzal: Inverse problem and equivalent contact systems. *Preprint, 2021*.

1. A relevant problem in mechanics is the so-called inverse problem: Given a second-order differential equation $\xi$ on $TQ$, when there exists a Lagrangian $L$ such that $\xi$ is the Euler-Lagrange vector field for $L$?
2. This problem has been investigatd by decades (the Helmholtz conditions have been interpreted geometrically using the geometry of tangent bundles).
3. Our research is focused on a similar problem for contact Lagrangian systems.

## Classification of contact Lagrangians

M. de León, I. Guti, M. Lainz-Valcárzal: Classification of contact Lagrangian systems.

1. The classification of Lagrangian systems has been studied by several authors in the autonomous and non–autonomous cases, in relation with the regularization problem.
2. In this ongoing paper we discuss similar problems for contact Lagrangian systems.

## Vakonomic contact systems

M. de León, M Lainz-Valcázar, M.C. Muñoz-Lecanda: The Herglotz principle and vakonomic dynamics. (To appear in Springer, 2021).
M. de León, M Lainz-Valcázar, M.C. Muñoz-Lecanda, N. Román-Roy: Constrained Lagrangian dissipative contact dynamic. (In preparation).

1. Nonholonomic and vakonomic dynamics obey to different principles. These two different approaches and their relations have been widely discussed in the last two decades.
2. Nonholonomic equations are related with mechanical systems subjected to constraints whilw vakonomic dynamics are related with control problems.
3. In the above papers we have started the discussion in the contact case.

## Tulczyjew triples in contact dynamics

O. Esen, M. de León, M Lainz-Valcázar, J.C. Marrero: Tulczyjew triples in contact dynamics. (Preprint 2021).

1. The construction of Tulczyjew triples (named after their creator, W.M. Tulczyjew) provides a global view of the Lagrangian and Hamiltonian dynamics as subvarieties of the so-called special symplectic varieties.
2. In this paper we develop the corresponding theory in the contact case, where Lagrangian submanifols are replaced by Legendrian submanifolds of special contact manifolds.