Ex 1 

$\xi$ SODE

$\xi$ vector field on $TQ$ /

$$S(\xi) = \Delta$$

If

$$\xi = X^i \frac{\partial}{\partial q^i} + Y^i \frac{\partial}{\partial \dot{q}^i},$$

then

$$\left.\begin{array}{l} S(\xi) = X^i \dfrac{\partial}{\partial \dot{q}^i} \\ \Delta = \dot{q}^i \dfrac{\partial}{\partial \dot{q}^i} \end{array}\right\} \Rightarrow X^i = \dot{q}^i$$

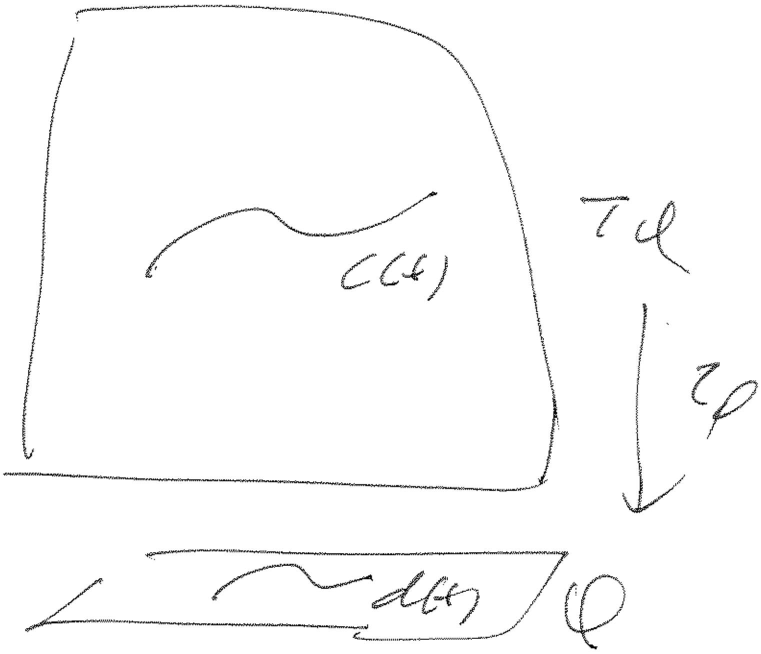
Therefore

$$\boxed{\xi \text{ SODE} \iff \text{locally } \xi = \dot{q}^i \frac{\partial}{\partial q^i} + Y^i \frac{\partial}{\partial \dot{q}^i}}$$

Let $c(t) = (q^i(t), \dot{q}^i(t))$ be an integral curve of $\xi$. Then

$$\left.\begin{array}{l} \dfrac{dq^i}{dt} = \dot{q}^i \\ \dfrac{d\dot{q}^i}{dt} = Y^i(q, \dot{q}) \end{array}\right\} \Rightarrow \boxed{\frac{d^2 q^i}{dt^2} = Y^i\left(q, \frac{dq}{dt}\right)}$$

Let $c(t) = (q^i(t), \dot{q}^i(t))$ an integral curve of $\xi$, and let $d(t) = \tau_Q \circ c(t)$ its projection to $Q$




The tangent lift of $d(t)$ is

$$\dot{d}(t) = \left(q^i(t), \frac{dq^i}{dt}\right)$$

But

$$c(t) = \left(q^i(t), \frac{dq^i}{dt}\right)$$

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# Ex 2



$$\begin{tikzcd} TQ \arrow[rr, "Leg"] \arrow[dr, "\tau_Q"'] & & T^*Q \arrow[dl, "\pi_Q"] \\ & Q & \end{tikzcd}$$

$$Leg(q^i, \dot{q}^i) = \left(q^i, p_i = \frac{\partial L}{\partial \dot{q}^i}\right)$$

$$v_q \in T_q Q \longmapsto Leg(v_q) \in T_q^* Q$$

$$\langle Leg(v_q), w_q \rangle = dL(v_q)(W_{v_q})$$

$$\forall w_q \in T_q Q, \quad W_{v_q} \in T_{v_q}(TQ)$$

$$T\tau_Q(W_{v_q}) = w_q$$

$$Leg(q^i, \dot{q}^i) = Leg\left(\dot{q}^i \frac{\partial}{\partial q^i}\right)$$

$$Leg\left(\dot{q}^i \frac{\partial}{\partial q^i}, w^i \frac{\partial}{\partial q^i}\right) = \left(\frac{\partial L}{\partial \dot{q}^r} dq^r\right)\left(w^j \frac{\partial}{\partial q^j} + -\right)$$

$$= \frac{\partial L}{\partial \dot{q}^i} \cdot w^i, \quad \forall w^i$$

Then

$$Leg(q^i, \dot{q}^i) = \left(q^i, \frac{\partial L}{\partial \dot{q}^i}\right)$$

1. $Leg^* \vartheta_0 = \vartheta_L$

$$\boxed{Leg^* \vartheta_0} = Leg^*(p_i \, dq^i)$$
$$= (p_i \circ Leg) \, d(q^i \circ Leg)$$
$$= \frac{\partial L}{\partial \dot{q}^i} \, dq^i = \boxed{\vartheta_L}$$

2. Then $\boxed{Leg^* \omega_0 = \omega_L}$

d and $Leg^*$ commute

If $H = E_L \circ Leg^{-1}$ ③

③ $TLeg(\xi_L) = X_H$

$i_{\xi_L}\omega_L = dE_L$

$i_{X_H}\omega_Q = dH$

$(i_{TLeg(\xi_L)}\omega_Q)(TLeg(Z))$

$= (Leg^*\omega_Q)(\xi_L, Z)$

$= \omega_L(\xi_L, Z)$

$= (i_{\xi_L}\omega_L)(Z)$

$= dE_L(Z)$

$= (Leg^* dH)(Z)$

$= dH(TLeg(Z)) \longrightarrow \boxed{TLeg(\xi_L) = X_H}$

L regular

$\Updownarrow$

Leg local diffeomorphism

$(Leg)_* = \begin{pmatrix} I & * \\ 0 & W_{ij} \end{pmatrix}$

We assume L hyperregular ($\Leftrightarrow$ Leg global diffeomorphism)

(a) An alternative definition

$$TQ \xrightarrow{L} \mathbb{R}$$

$$Leg(v_q) = dL_{|T_qQ}(v_q) \in T_q^*Q$$

$$T_qQ \xrightarrow{L_{|T_qQ}} \mathbb{R}$$

$$L_{|T_qQ}(\dot q^i) = L(q^i, \dot q^i)$$

fixed

$\Downarrow$

the differential is just $\frac{\partial L}{\partial \dot q^i}$

$$T_{v_q}(T_qQ) \xrightarrow{d(L_{|T_qQ})} \mathbb{R}$$

$$\begin{array}{c} T_{v_q}(T_qQ) \\ \parallel\!\!\wr \\ T_qQ \end{array} \qquad d(L_{|T_qQ}) \text{ “}\in\text{” } T_q^*Q$$

The tangent space of a vector space $V$ at a point $x \in V$ can be identified to $V$

$$T_xV \equiv V$$

Then

$$Leg(q^i, \dot q^i) = \left(q^i, \frac{\partial L}{\partial \dot q^i}\right)$$

5 Let $H : T^*Q \longrightarrow \mathbb{R}$

We can define a mapping

$$\begin{tikzcd} T^*Q \arrow[rr, "\widetilde{Leg}"] \arrow[dr, "\pi_Q"'] & & TQ \arrow[dl, "\tau_Q"] \\ & Q & \end{tikzcd}$$

as follows

$$\widetilde{Leg}(\alpha_q) = d(H|_{T_q^*Q})(\alpha_q) : T_q^*Q \longrightarrow \mathbb{R}$$

$\Rightarrow d(H|_{T_q^*Q})$ can be identified to an element in $(T_q^*Q)^* \cong T_qQ$

In local coordinates

$$\boxed{\widetilde{Leg}(q^i, p_i) = \left(q^i, \frac{\partial H}{\partial p_i}\right)}$$

This is a particular case of the so-called fiber derivative.

Ex 4

(2)

$$\bar{\delta}(X_H) = dH - (R(H)+H)\eta$$
$$\shortparallel$$
$$i_{X_H} d\eta + \eta(X_H)\,\eta$$

(1) Apply to $R$ (both sides)

$$dH(R) - (R(H)+H)\overbrace{\eta(R)}^{1}$$
$$\shortparallel$$
$$\underbrace{(i_{X_H} d\eta)(R)}_{0} + \eta(X_H)\underbrace{\eta(R)}_{1}$$

$$R(H) - (R(H)+H) = \eta(X_H)$$
$$\Rightarrow \boxed{\eta(X_H) = -H}$$

(2) $$\boxed{\mathcal{L}_{X_H}\eta} = i_{X_H} d\eta + d\, i_{X_H}\eta$$
$$= dH - (R(H)+H)\eta - \overbrace{\eta(X_H)}^{-H}\eta + d(\overbrace{\eta(X_H)}^{-H})$$
$$= \cancel{dH} - R(H)\eta \cancel{- H\eta} + \cancel{H\eta} - \cancel{dH}$$
$$= \boxed{-R(H)\eta}$$

and conversely

## Ex 5

## Characteristic distributions $\mathcal{C}$

Symplectic manifolds $(M, \omega)$

$$\mathcal{C} = TM$$

Cosymplectic manifolds $(M, \Omega, \eta)$

$$\mathcal{C} = \operatorname{Ker} \eta$$

(2n)

Locally conformal symplectic manifolds $(M, \Omega, \theta)$

$$\mathcal{C} = TM$$

Contact manifolds

$$\mathcal{C} = TM$$

$(M, \omega)$ symplectic

$$\Downarrow$$

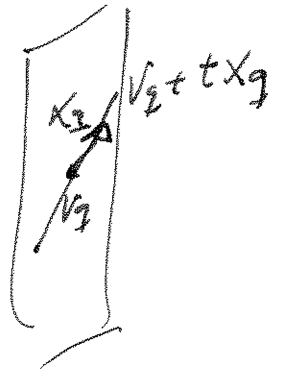
$(TM, \omega^c)$ symplectic

$$\begin{array}{ccc} TQ & & X^V, X^C \\ \downarrow \tau_Q & & \uparrow \\ Q & & X \end{array}$$

$X^C \leftrightarrow T\varphi_t$, if $X \leftrightarrow \varphi_t$

$X^V(v_q)$ = tangent vector to the curve

$t \longmapsto v_q + t X_q$




$\cdot\ q$

If $X = X^i \dfrac{\partial}{\partial q^i}$

$X^V = X^i \dfrac{\partial}{\partial \dot q^i}$

$X^C = X^i \dfrac{\partial}{\partial q^i} + \dot q^j \dfrac{\partial X^i}{\partial q^j} \dfrac{\partial}{\partial \dot q^i}$

$$\omega^c(X^c, Y^c) = (\omega(X,Y))^c$$

$$\omega^c(X^c, Y^V) = (\omega(X,Y))^V$$

$$\omega^c(X^V, Y^V) = 0$$

$$\begin{tikzcd} TQ \arrow[d, "\tau_Q"'] \arrow[dr, bend left, "f^c \equiv df"] & \\ Q \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

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$$(M, \omega, H) \xrightarrow{X_H} (TM, \omega^c)$$

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$$

$$\mathrm{Im}\, X_H = (q^i, p_i;\ \underbrace{\frac{\partial H}{\partial p_i}}_{\dot q^i}, \underbrace{-\frac{\partial H}{\partial q^i}}_{\dot p_i})$$

$$\omega = dq^i \wedge dp_i$$

$$\omega^c = d\dot q^i \wedge dp_i + dq^i \wedge d\dot p_i$$

$$X_H^*(\omega^c) = 0$$

$\dim \mathrm{Im}\, X_H = 2n$

$\dim TM = 4n$

Ex 7

(1)

$(M, \eta)$ contact

$N \subset M$ legendrian

$N \equiv \phi_a = 0$

$TN = \{X \mid X(\phi_a) = 0\}$

$TN^0 = \langle d\phi_a \rangle$

$TN^{\perp} = \#_\Lambda(TN^0) = \langle Z_a = \#_\Lambda(d\phi_a) \rangle$

$$Z_a = \left( \frac{\partial \phi_a}{\partial q^i} + p_i \frac{\partial \phi_a}{\partial z} \right) \frac{\partial}{\partial p_i} + \frac{\partial \phi_a}{\partial p_i} \left( \frac{\partial}{\partial q^i} - p_i \frac{\partial}{\partial z} \right)$$

Coisotropic if $TN^{\perp} \subset TN$

$\Leftrightarrow Z_a(\phi_b) = 0 \quad \forall a, b$

$$\boxed{\left( \frac{\partial \phi_a}{\partial q^i} + p_i \frac{\partial \phi_a}{\partial z} \right) \frac{\partial \phi_b}{\partial p_i} + \frac{\partial \phi_a}{\partial p_i} \left( \frac{\partial \phi_b}{\partial q^i} - p_i \frac{\partial \phi_b}{\partial z} \right) = 0 \quad \forall a, b}$$

② 

1) $TN^{\perp} \subset TN$

If $TN^{\perp} = TN$, then

$\dim N = 2n - r \quad 1 \le a \le r$

$TN^{\perp} = \langle Z_a \rangle$ rank $r$

If $TN^{\perp} = TN$, then $\boxed{n = r}$

$r = 2n - r$

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2) $\eta|_{TN} = 0$

$TN = TN^{\perp} = \#_1(TN^0)$

$\eta(v) = \eta(\#_1(\alpha))$, $\alpha \in TN^0$

$= \Lambda(\eta, \alpha) = 0$

$\Leftrightarrow R$

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